Debye screening and the Meissner effect in a two-flavor color superconductor

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I compute the gluon self-energy in a color superconductor with two flavors of massless quarks, where condensation of Cooper pairs breaks $SU(3)_c$ to $SU(2)_c$. At zero temperature, there is neither Debye screening nor a Meissner effect for the three gluons of the unbroken $SU(2)_c$ subgroup. The remaining five gluons attain an electric as well as a magnetic mass. For temperatures approaching the critical temperature for the onset of color superconductivity, or for gluon momenta much larger than the color-superconducting gap, the self-energy assumes the form given by the standard hard-dense loop approximation. The gluon self-energy determines the coefficient of the kinetic term in the effective low-energy theory for the condensate fields.

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I. INTRODUCTION

Single-gluon exchange between two quarks is attractive in the color antitriplet channel. Therefore, sufficiently cold and dense quark matter is a color superconductor [1,2].

In some aspects, color superconductivity is similar to ordinary (BCS) superconductivity [3,4]. For instance, like electrons in a BCS superconductor, quarks form Cooper pairs. At zero temperature, $T=0$, the ground state of the system is no longer a Fermi sea of quarks (and a Dirac sea of antiquarks), but a Bose condensate of quark Cooper pairs. In the normal phase the excitation of a particle-hole pair at the Fermi surface costs no energy. In the superconducting phase, however, exciting a quasiparticle–quasiparticle-hole pair costs at least an energy $2\phi_0$, where ϕ_0 is the zero-temperature gap. Another similarity between color and BCS superconductivity is that, in weak coupling, the critical temperature T_c for ''melting'' the Cooper pair condensate is $T_c \simeq 0.57\phi_0$ [5,6].

There are, however, also fundamental differences between color and BCS superconductivity. First of all, a BCS superconductor requires the presence of an atomic lattice with phonons that cause electrons to form Cooper pairs. On the other hand, in QCD gluons themselves cause quarks to condense. Another difference is that in BCS theory the zero-temperature gap depends on the BCS coupling constant G as $\phi_0 \sim \mu \exp(-c_{\rm BCS}/G^2)$ [3,4], where μ is the chemical potential, and $c_{\rm BCS}=$const, while in a color superconductor, $\phi_0 \sim \mu \exp(-c_{\rm QCD}/g)$ [7,8], where g is the QCD coupling constant, and $c_{\rm BCS} \neq c_{\rm QCD}=$const.

The physical reason for the change in the parametric dependence on the coupling constant is that, because gluons are massless, gluon-mediated interactions are long-range, in contrast with BCS theory, where phonon exchange is typically assumed to be a pointlike interaction [3,4]. The long-range nature of gluon exchange manifests itself in the infrared singular behavior of the gluon propagator. This enhances the contribution of very soft, collinear gluons in the gap equations [5,6], and causes the $1/g$ in the exponent, instead of a $1/g^2$ which would appear if gluons were massive [9], or gluon exchange a pointlike interaction as assumed in Nambu–Jona-Lasinio-type approaches to color superconductivity [2].

Some care has to be taken in determining the coefficient $c_{\rm QCD}$. This constant differs when one uses the free gluon propagator [10] in the solution of the gap equations instead of a propagator which takes into account the presence of the cold and dense quark medium. By now, several authors [5,6,11–14] have confirmed Son's original result $c_{\rm QCD} = 3\pi^2/\sqrt{2}$ [8], obtained by using the gluon propagator in the so-called ''hard-dense-loop'' (HDL) limit [15,16]. The gluon propagator in the HDL limit is obtained by resummation of the gluon self-energy, computed to one-loop order for gluon energies p_0 and momenta p that are much smaller than the quark chemical potential μ.

In weak coupling, the temperatures where quark matter is color-superconducting are much smaller than the quark chemical potential, $T \sim \phi_0 \sim \mu \exp(-c_{\rm QCD}/g) \ll \mu$. Therefore, to leading order the contributions of gluon and ghost loops to the one-loop gluon self-energy can be neglected, and the main contribution comes from the quark loop. This is very similar to ordinary superconductivity, where the one-loop photon self-energy is determined by an electron loop.

In the standard HDL approximation, however, the quark excitations in the loop are considered to be those of the normal and not of the superconducting phase. This is in principle inconsistent. The aim of the present work is to amend this shortcoming and to compute the gluon self-energy in the color-superconducting phase.

For the sake of definiteness, I consider a color superconductor with $N_f=2$ flavors of massless quarks, and assume that quarks condense in a channel with total spin $J=0$ and even parity. In this case, the quark-quark condensate breaks $SU(3)_c$ to $SU(2)_c$. Consequently, one expects that the three gluons of the unbroken $SU(2)_c$ subgroup remain massless, while the other five gluons of the original $SU(3)_c$ obtain masses through the Anderson-Higgs mechanism. It is therefore necessary to consider different gluon colors separately.

I derive a general expression for the quark contribution to the gluon self-energy, and study the limit where the gluon energy $p_0=0$ and the gluon momentum $p \to 0$. For electric gluons, this limit gives the Debye mass, while for magnetic gluons, it gives the Meissner mass. I also consider the limit where $p_0=0$, but $p \gg \phi_0$. In this case, the gluon momentum

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is large enough to resolve individual quarks in a Cooper pair; consequently, the Debye masses approach their values in the normal phase and the Meissner effect vanishes.

Debye screening of static color-electric fields and the Meissner effect for static color-magnetic fields are in principle quite analogous to Debye screening and the Meissner effect for electromagnetic fields in ordinary superconductors [3,4]. However, the somewhat more complicated color and flavor structure of a quark-quark condensate in comparison to an electron-electron condensate gives rise to an additional degree of complexity. While studying these effects in a color superconductor is interesting in itself, they might have, however, far greater implications for color superconductivity than the corresponding effects in ordinary superconductors: unlike photons, gluons themselves are responsible for condensation of quark pairs. The modification of the gluon self-energy in the superconducting phase directly enters the gap equation through the gluon propagator, and so might change the value for the gap. On the other hand, the influence of the photon self-energy on electron condensation is at best a higher order effect.

Although effects from quark condensation in the gluon propagator vanish for large gluon energies and momenta, one can *a priori* not exclude that they will not change the solution of the gap equations. For instance, to assess the importance of the Meissner effect, note that, in the HDL approximation, the main contribution to the gap equations comes from color-magnetic fields with momenta $p \sim (m_g^2\, \phi_0)^{1/3} \gg \phi_0$, where m_g is the gluon mass [5,6,8,12]. As will be seen below, the Meissner effect is small, but not absent, at the same momentum scale. This means that the Meissner effect can indeed influence the solution of the gap equation. A first estimate of this effect (neglecting the color-flavor structure of the condensate and considering only the dominant contribution to the gluon self-energy) was given in [17], and a reduction of the zero-temperature gap was found.

This paper is organized as follows. In Sec. II a compact derivation of the quark contribution to the gluon self-energy is presented, mainly to introduce the notation and the concept of Nambu-Gor'kov spinors [4], which considerably simplify calculations at nonzero chemical potential. In Sec. III the quark contribution to the gluon self-energy is explicitly computed in the normal phase. The HDL limit is derived to show that the Nambu-Gor'kov method indeed gives the correct answer. Section IV generalizes the previous results to the superconducting phase. In Sec. V, the zero-energy, zero-momentum limit of the gluon self-energy is studied, which yields the Debye as well as the Meissner masses in the superconducting phase. Section VI discusses how, for nonzero gluon momenta $p \gg \phi_0$, the Debye masses approach their values in the normal phase, and the Meissner effect vanishes. Readers not interested in technical details should skip Secs. II to VI and move on to Sec. VII, where the main results of this work are summarized, conclusions are drawn, and an outlook for future studies is given.

I use natural units, $\hbar = c = k_B = 1$, and work in Euclidean space-time $\mathbf{R}^4 \equiv V/T$, where V is the volume and T the temperature of the system. Nevertheless, I find it convenient to retain the Minkowski notation for 4-vectors, with a metric tensor $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$. For instance, the space-time coordinate vector is $x^\mu \equiv (t, \mathbf{x})$, $t \equiv -i\tau$, where τ is Euclidean time. 4-momenta are denoted as $K^\mu \equiv (k_0, \mathbf{k})$, $k_0 \equiv -i\omega_n$, where ω_n is the Matsubara frequency, $\omega_n \equiv 2n\pi T$ for bosons and $\omega_n \equiv (2n+1)\pi T$ for fermions, $n = 0, \pm 1, \pm 2, \ldots$. The absolute value of the 3-momentum $\mathbf{k}$ is denoted as $k \equiv |\mathbf{k}|$, and its direction as $\hat{\mathbf{k}} \equiv \mathbf{k}/k$.

II. THE GENERATING FUNCTIONAL AT NONZERO CHEMICAL POTENTIAL

Consider QCD with N_f quark flavors, at nonzero chemical potential. The $N_f \times N_f$ matrix of quark masses m_f will be denoted as $m \equiv \mathrm{diag}(m_1, m_2, \ldots, m_{N_f})$. Let us consider a color neutral system, i.e., there is no chemical potential for color, however, there can be in general a chemical potential μ_f for each quark flavor f. Let us denote the $N_f \times N_f$ chemical potential matrix as $\mu \equiv \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{N_f})$. Then, the generating functional for the N-point functions of the theory reads (normalization factors are suppressed)

$$\mathcal{Z}[J, \bar{\eta}, \eta] = \int \mathcal{D}U[A] \exp\left[\int_x (\mathcal{L}_A + J^a_\mu A^\mu_a)\right] \mathcal{Z}[A, \bar{\eta}, \eta], \tag{1a}$$

$$\mathcal{Z}[A, \bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \exp\left\{\int_x [\bar{\psi}(i\gamma^\mu \partial_\mu + \mu\gamma_0 - m + g\gamma^\mu A^a_\mu T_a)\psi + \bar{\eta}\psi + \bar{\psi}\eta]\right\}. \tag{1b}$$

Here, $\mathcal{D}U[A]$ is the gauge invariant measure for the integration over the gauge fields A^μ_a. The space-time integration is defined as $\int_x \equiv \int_0^{1/T} d\tau \int_V d^3\mathbf{x}$. g is the QCD coupling constant, γ^μ are the Dirac matrices, and $T_a = \lambda_a/2$ the generators of $SU(N_c)$; for QCD, $N_c = 3$, and λ_a are the Gell-Mann matrices. The quark fields ψ (as well as the external fields η) are $4N_cN_f$-component spinors, i.e., they carry Dirac indices $\alpha = 1, \ldots, 4$, fundamental color indices $i = 1, \ldots, N_c$, and flavor indices $f = 1, \ldots, N_f$. The Lagrangian for the gauge fields consists in general of three parts,

$$\mathcal{L}_A = \mathcal{L}_F + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{FPG}, \tag{2}$$

where

$$\mathcal{L}_F = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} \tag{3}$$

is the gauge field part, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ is the field strength tensor. The parts corresponding to gauge fixing, $\mathcal{L}_{\mathrm{gf}}$, and to Fadeev-Popov ghosts, $\mathcal{L}_{FPG}$, need not be specified: it will be seen that they are inconsequential for the following.

In the vacuum, the ground state of the system consists of the Dirac sea, i.e., all negative energy (antiquark) states are occupied, while all positive energy (quark) states are empty. At zero temperature and nonzero chemical potential, $\mu_f > 0$, however, the ground state consists of the Dirac sea *and* the Fermi sea, i.e., positive energy states which are occupied up to the Fermi energy μ_f. Formally, this is expressed by the term $\bar{\psi}\mu\gamma_0\psi$ in the generating functional (1b), which ensures that the energy of excited states of flavor f is measured with respect to the Fermi energy μ_f, and not with respect to the vacuum at zero density.

This shift of the energy scale introduces an apparent asymmetry. One can restore the symmetry by the following trick. Introduce M identical copies (''replicas'') of the original quark fields. All copies are supposed to interact with the gluon field in the same way. At the end, after having computed N-point functions for this extended system, M will be set equal to 1. The generating functional (1b) for the quark part is replaced by

$$\mathcal{Z}[A,\bar{\eta},\eta]\to\mathcal{Z}_M[A,\bar{\eta},\eta]\equiv(\mathcal{Z}[A,\bar{\eta},\eta])^M. \tag{4}$$

Now define the charge conjugate spinors ψ_C, $\bar{\psi}_C$ through

$$\psi\equiv C\,\bar{\psi}_C^T, \quad \bar{\psi}\equiv\psi_C^T C, \tag{5}$$

where $C\equiv i\gamma^2\gamma_0$ is the charge conjugation matrix; $C=-C^{-1}=-C^T=-C^\dagger$. In half of the M copies in Eq. (4), replace $\bar{\psi}$, ψ by the charge conjugate spinors $\bar{\psi}_C$, ψ_C. Using $C\gamma_\mu C^{-1}=-\gamma_\mu^T$, and the anticommutation property of the (Grassmann-valued) quark spinors, one obtains after an integration by parts (and disregarding the overall normalization)

$$\begin{aligned}\mathcal{Z}_M[A,\bar{\eta},\eta,\bar{\eta}_C,\eta_C] &= \Bigg(\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}_C\,\mathcal{D}\psi_C \\ &\quad\times\exp\Bigg\{\int_x[\bar{\psi}(i\gamma^\mu\partial_\mu+\mu\gamma_0-m+gA_\mu^a\Gamma_a^\mu)\psi \\ &\quad+\bar{\psi}_C(i\gamma^\mu\partial_\mu-\mu\gamma_0-m+gA_\mu^a\bar{\Gamma}_a^\mu)\psi_C \\ &\quad+\bar{\eta}\psi+\bar{\psi}\eta+\bar{\eta}_C\psi_C+\bar{\psi}_C\eta_C]\Bigg\}\Bigg)^{M/2}.\end{aligned} \tag{6}$$

Here,

$$\Gamma_a^\mu\equiv\gamma^\mu T_a, \quad \bar{\Gamma}_a^\mu\equiv C(\gamma^\mu)^T C^{-1}T_a^T\equiv-\gamma^\mu T_a^T, \tag{7}$$

and charge conjugate external fields $\bar{\eta}_C$ and η_C were defined analogous to Eq. (5). Let us now introduce the $8N_cN_f$-component (Nambu-Gor'kov) spinors

$$\Psi\equiv\begin{pmatrix}\psi\\ \psi_C\end{pmatrix}, \quad \bar{\Psi}\equiv(\bar{\psi},\ \bar{\psi}_C), \quad H\equiv\begin{pmatrix}\eta\\ \eta_C\end{pmatrix}, \quad \bar{H}\equiv(\bar{\eta},\ \bar{\eta}_C), \tag{8}$$

and the $8N_cN_f\times 8N_cN_f$-dimensional inverse propagator

$$\mathcal{S}_0^{-1}(x,y)\equiv\begin{pmatrix}[G_0^+]^{-1}(x,y) & 0\\ 0 & [G_0^-]^{-1}(x,y)\end{pmatrix}, \tag{9}$$

where

$$[G_0^\pm]^{-1}(x,y)\equiv-i(i\gamma_\mu\partial_x^\mu\pm\mu\gamma_0-m)\delta^{(4)}(x-y) \tag{10}$$

is the inverse propagator for non-interacting quarks (upper sign) or charge conjugate quarks (lower sign), respectively. Furthermore, denote

$$\hat{\Gamma}_a^\mu\equiv\begin{pmatrix}\Gamma_a^\mu & 0\\ 0 & \bar{\Gamma}_a^\mu\end{pmatrix}. \tag{11}$$

Then, the generating functional (6) can be written in the compact form

$$\begin{aligned}\mathcal{Z}_M[A,\bar{H},H] &= \int\prod_{r=1}^{M/2}\mathcal{D}\bar{\Psi}_r\,\mathcal{D}\Psi_r\exp\Bigg\{\sum_{r=1}^{M/2}\Bigg[\int_{x,y}\bar{\Psi}_r(x)\,\mathcal{S}_0^{-1}(x,y)\,\Psi_r(y) \\ &\quad+\int_x(g\,\bar{\Psi}_r A_\mu^a\hat{\Gamma}_a^\mu\Psi_r+\bar{H}_r\Psi_r+\bar{\Psi}_r H_r)\Bigg]\Bigg\}.\end{aligned} \tag{12}$$

In this form, all reference to the chemical potentials μ_f has been absorbed in the inverse propagator (9). Therefore, the generating functional for QCD, Eq. (1a) with Eq. (12), is formally identical to that at zero chemical potential. The apparent asymmetry introduced by a nonzero chemical potential μ_f has been restored by the introduction of charge conjugate fields; the associated charge conjugate propagator G_0^- appears on equal footing with the ordinary propagator G_0^+.

III. THE GLUON SELF-ENERGY IN THE NORMAL PHASE

The gluon self-energy is defined as

$$\Pi\equiv\Delta^{-1}-\Delta_0^{-1}, \tag{13}$$

where Δ^{-1} is the resummed and Δ_0^{-1} the free inverse gluon propagator; for instance, in momentum space and in covariant gauge,

$$[\Delta_0^{-1}]_{ab}^{\mu\nu}(P)=\delta_{ab}\left(P^2g^{\mu\nu}+\frac{1-\alpha}{\alpha}P^\mu P^\nu\right). \tag{14}$$

To one-loop order, the gluon self-energy receives contributions from gluon loops (through the 3-gluon and 4-gluon vertices), ghost loops (through the ghost-gluon vertex), and quark loops (through the quark-gluon vertex),

$$\Pi=\Pi_g+\Pi_{FPG}+\Pi_q+O(g^3). \tag{15}$$

Π_g and Π_{FPG} are independent of μ, effects from nonzero chemical potential enter only through Π_q. For dimensional reasons,

$$\Pi_g,\ \Pi_{FPG}\sim g^2T^2, \quad \Pi_q\sim g^2(\mu^2+a\,T^2), \tag{16}$$

with some constant a.

The superconducting condensate melts when the temperature T exceeds the critical temperature $T_c \simeq 0.57\phi_0$ [5,6], where ϕ_0 is the magnitude of the superconducting gap at $T=0$. In weak coupling QCD, $\phi_0 \sim \mu \exp(-c_{\rm QCD}/g) \ll \mu$ [5–8,11–14], and temperature effects can be neglected to leading order. This means that, for the temperatures of interest in this work, one can neglect the contributions from gluon and ghost loops to the gluon self-energy, and consider the quark contribution only, $\Pi \simeq \Pi_q$.

Due to the aforementioned similarity between the generating functional (1a), with the quark part (12), and the one at zero chemical potential, it is not difficult to derive the quark contribution to the one-loop gluon self-energy. If there is no superconducting condensate, this contribution is

$$\Pi_{0ab}^{\mu\nu}(x,y) \equiv \frac{M}{2} g^2 \, {\rm Tr}_{s,c,f,NG}[\hat{\Gamma}_a^\mu \, \mathcal{S}_0(x,y) \, \hat{\Gamma}_b^\nu \, \mathcal{S}_0(y,x)]. \quad (17)$$

Here, the factor $M/2$ arises from the fact that there are $M/2$ identical species of quarks described by spinors Ψ_r in Eq. (12), which contribute to the gluon self-energy. In the following, set $M=1$, to recover the original theory. The trace in Eq. (17) is taken over 4-dimensional spinor space, N_c-dimensional color space, N_f-dimensional flavor space, and the 2-dimensional space of regular and charge-conjugate spinors (Nambu-Gor'kov space).

In the following, the self-energy (17) is evaluated in momentum space. Use will be made of translational invariance, $\mathcal{S}_0(x,y) \equiv \mathcal{S}_0(x-y)$, cf. Eq. (10), and of the Fourier transforms

$$\mathcal{S}_0(x) = \frac{T}{V} \sum_K \, e^{-iK\cdot x} \, \mathcal{S}_0(K), \quad (18a)$$

$$-i\,\delta^{(4)}(x) \equiv \delta^{(3)}(\mathbf{x}) \, \delta(\tau) = \frac{T}{V} \sum_K \, e^{-iK\cdot x}, \quad (18b)$$

$$\int_x e^{iK\cdot x} = \frac{V}{T} \, \delta^{(4)}_{K,0}, \quad (18c)$$

where $\Sigma_K \equiv \Sigma_n V \int d^3\mathbf{k}/(2\pi)^3$. Here, the quark propagator in momentum space is

$$\mathcal{S}_0(K) \equiv \begin{pmatrix} G_0^+(K) & 0 \\ 0 & G_0^-(K) \end{pmatrix},$$

$$G_0^\pm(K) \equiv (\gamma^\mu K_\mu \pm \mu\gamma_0 - m)^{-1}. \quad (19)$$

Then, the gluon self-energy in momentum space is

$$\Pi_{0ab}^{\mu\nu}(P) = \frac{1}{2} g^2 \frac{T}{V} \sum_K \, {\rm Tr}_{s,c,f,NG}[\hat{\Gamma}_a^\mu \, \mathcal{S}_0(K) \, \hat{\Gamma}_b^\nu \, \mathcal{S}_0(K-P)]. \quad (20)$$

As a warm-up exercise, and to confirm that the method of the Nambu-Gor'kov propagators indeed gives the correct answer, let us derive from Eq. (20) the standard hard-dense-loop (HDL) result [15,16] for the quark contribution to the gluon self-energy. To see the analogy to the computation in the superconducting phase, cf. Sec. IV, Eq. (20) will be evaluated in several steps.

A. Trace over Nambu-Gor'kov space

First perform the trace over Nambu-Gor'kov space. With Eqs. (11) and (19), one obtains

$$\Pi_{0ab}^{\mu\nu}(P) = \frac{1}{2} g^2 \frac{T}{V} \sum_K \, {\rm Tr}_{s,c,f}[\Gamma_a^\mu \, G_0^+(K) \, \Gamma_b^\nu \, G_0^+(K-P) + \bar{\Gamma}_a^\mu \, G_0^-(K) \, \bar{\Gamma}_b^\nu \, G_0^-(K-P)]. \quad (21)$$

B. Trace over flavor space

The vertices Γ_a^μ and $\bar{\Gamma}_a^\mu$ are diagonal in flavor space,

$$(\Gamma_a^\mu)_{fg} = \delta_{fg} \, \Gamma_a^\mu, \quad (\bar{\Gamma}_a^\mu)_{fg} = \delta_{fg} \, \bar{\Gamma}_a^\mu. \quad (22)$$

The free propagators $G_0^\pm$ are also diagonal in flavor space, but for $\mu_f \neq \mu_g$, $f \neq g$, $f,g \in \{1, \ldots, N_f\}$, the diagonal components are in general not equal. To proceed, assume that all chemical potentials are equal, $\mu_1 = \mu_2 = \ldots = \mu_{N_f} \equiv \mu$, such that

$$(G_0^\pm)_{fg} = \delta_{fg} \, G_0^\pm. \quad (23)$$

(For notational convenience, I am somewhat sloppy with indices here and throughout the rest of the paper: I use the same symbol, $G_0^\pm$, for the $4N_cN_f \times 4N_cN_f$ matrix on the left-hand side of this equation and for the $4N_c \times 4N_c$ matrix on the right-hand side.) Thus, the trace over flavor space simply gives a factor N_f,

$$\Pi_{0ab}^{\mu\nu}(P) = \frac{1}{2} g^2 N_f \, \frac{T}{V} \sum_K \, {\rm Tr}_{s,c}[\Gamma_a^\mu \, G_0^+(K) \, \Gamma_b^\nu \, G_0^+(K-P) + \bar{\Gamma}_a^\mu \, G_0^-(K) \, \bar{\Gamma}_b^\nu \, G_0^-(K-P)]. \quad (24)$$

This expression is easily generalized to the case where the chemical potentials are not equal. Then, instead of the prefactor N_f one would have a sum over flavors f, where the value of the chemical potential in the propagators $G_0^\pm$ in the fth term of the sum is equal to μ_f.

C. Trace over color space

The free quark propagator is diagonal in (fundamental) color space,

$$(G_0^\pm)_{ij} = \delta_{ij} \, G_0^\pm. \quad (25)$$

The only nontrivial color structure thus arises from the generators of $SU(3)_c$. On account of

$${\rm Tr}_c(T_aT_b) = {\rm Tr}_c(T_aT_b)^T = {\rm Tr}_c(T_a^TT_b^T) = \frac{1}{2}\,\delta_{ab}, \quad (26)$$

one obtains

$$\Pi_{0ab}^{\mu\nu}(P)=\delta_{ab}\,\Pi_0^{\mu\nu}(P), \tag{27a}$$

$$\Pi_0^{\mu\nu}(P)=\frac{1}{4}g^2 N_f\,\frac{T}{V}\sum_K \mathrm{Tr}_s[\gamma^\mu G_0^+(K)\,\gamma^\nu G_0^+(K-P) + \gamma^\mu G_0^-(K)\,\gamma^\nu G_0^-(K-P)]. \tag{27b}$$

D. Mixed representations for the quark propagators

To proceed, let us assume that the quarks are massless, $m=0$. Then, write the quark propagator as

$$G_0^\pm(K)=\sum_{e=\pm}\frac{k_0\mp(\mu-ek)}{k_0^2-[\epsilon_{\mathbf{k}0}^e]^2}\Lambda_{\mathbf{k}}^{\pm e}\,\gamma_0, \tag{28}$$

where

$$\epsilon_{\mathbf{k}0}^e\equiv|\mu-ek|, \tag{29}$$

and

$$\Lambda_{\mathbf{k}}^e\equiv\frac{1}{2}(1+e\gamma_0\,\boldsymbol{\gamma}\cdot\hat{\mathbf{k}}) \tag{30}$$

are projectors onto states of positive ($e=+$) or negative ($e=-1$) energy. Now introduce a mixed representation for the quark propagator,

$$G_0^\pm(\tau,\mathbf{k})\equiv T\sum_{k_0} e^{-k_0\tau}G_0^\pm(K),$$

$$G_0^\pm(K)\equiv\int_0^{1/T}d\tau\, e^{k_0\tau}G_0^\pm(\tau,\mathbf{k}). \tag{31}$$

After performing the Matsubara sum in terms of a contour integral in the complex k_0 plane, one obtains

$$G_0^+(\tau,\mathbf{k})=-\sum_{e=\pm}\Lambda_{\mathbf{k}}^e\,\gamma_0\{(1-n_{\mathbf{k}0}^e)[\theta(\tau)-N(\epsilon_{\mathbf{k}0}^e)]\times\exp(-\epsilon_{\mathbf{k}0}^e\tau)-n_{\mathbf{k}0}^e[\theta(-\tau)-N(\epsilon_{\mathbf{k}0}^e)]\times\exp(\epsilon_{\mathbf{k}0}^e\tau)\}, \tag{32a}$$

$$G_0^-(\tau,\mathbf{k})=-\sum_{e=\pm}\gamma_0\,\Lambda_{\mathbf{k}}^e\{n_{\mathbf{k}0}^e[\theta(\tau)-N(\epsilon_{\mathbf{k}0}^e)]\times\exp(-\epsilon_{\mathbf{k}0}^e\tau)-(1-n_{\mathbf{k}0}^e)[\theta(-\tau)-N(\epsilon_{\mathbf{k}0}^e)]\times\exp(\epsilon_{\mathbf{k}0}^e\tau)\}. \tag{32b}$$

Here, $N(x)\equiv(e^{x/T}+1)^{-1}$, and

$$n_{\mathbf{k}0}^e\equiv\frac{\epsilon_{\mathbf{k}0}^e+\mu-ek}{2\,\epsilon_{\mathbf{k}0}^e} \tag{33}$$

are the occupation numbers of particles ($e=+1$) or antiparticles ($e=-1$) at zero temperature. Consequently, $1-n_{\mathbf{k}0}^e$ are the occupation numbers for particle-holes or antiparticle-holes.

Note that

$$G_0^\pm(-\tau,\mathbf{k})=-\gamma_0\,G_0^\mp(\tau,\mathbf{k})\,\gamma_0. \tag{34}$$

For $0\leqslant\tau\leqslant 1/T$, one derives with $1-N(x)=N(x)\,e^{x/T}$

$$G_0^\pm\left(\frac{1}{T}-\tau,\mathbf{k}\right)=-G_0^\pm(-\tau,\mathbf{k}), \tag{35}$$

the well-known Kubo-Martin-Schwinger relation for fermions [15].

Using the fact that $n_{\mathbf{k}0}^e\equiv\theta(\mu-ek)$, and $N(x)=1-N(-x)$, the propagators (32a),(32b) can be cast into the more familiar form

$$G_0^+(\tau,\mathbf{k})=-\Lambda_{\mathbf{k}}^+\,\gamma_0[\theta(\tau)-N_F^+(k)]\,e^{-(k-\mu)\tau}+\Lambda_{\mathbf{k}}^-\,\gamma_0[\theta(-\tau)-N_F^-(k)]\,e^{(k+\mu)\tau}, \tag{36a}$$

$$G_0^-(\tau,\mathbf{k})=\gamma_0\,\Lambda_{\mathbf{k}}^+[\theta(-\tau)-N_F^+(k)]\,e^{(k-\mu)\tau}-\gamma_0\,\Lambda_{\mathbf{k}}^-[\theta(\tau)-N_F^-(k)]\,e^{-(k+\mu)\tau}, \tag{36b}$$

where $N_F^\pm(k)\equiv N(k\mp\mu)$ is the Fermi–Dirac distribution function for particles (antiparticles). However, in view of the application to the superconducting phase in Sec. IV, it is advantageous to continue to use the form (32a),(32b).

Denoting $K_1\equiv K$ and $K_2\equiv K-P$, one computes the expressions

$$T\sum_{k_0}\mathrm{Tr}_s[\gamma^\mu G_0^\pm(K_1)\,\gamma^\nu G_0^\pm(K_2)] = T\sum_{k_0}\int_0^{1/T}d\tau_1\,d\tau_2\,e^{k_0\tau_1+(k_0-p_0)\tau_2}\times\mathrm{Tr}_s[\gamma^\mu G_0^\pm(\tau_1,\mathbf{k}_1)\,\gamma^\nu G_0^\pm(\tau_2,\mathbf{k}_2)] \tag{37}$$

as follows. To perform the Matsubara sum over k_0, use the identity [15]

$$T\sum_n e^{k_0\tau}=\sum_{m=-\infty}^{\infty}(-1)^m\,\delta\left(\tau-\frac{m}{T}\right), \tag{38}$$

valid for fermionic Matsubara frequencies, $k_0=-i(2n+1)\pi T$. Since $0\leqslant\tau_1,\,\tau_2\leqslant 1/T$ in Eq. (37), the delta function in Eq. (38) has support only for $m=1$, i.e., $\tau_2=1/T-\tau_1$. With the help of Eqs. (34) and (35), as well as $e^{p_0/T}=1$ for bosonic Matsubara frequencies $p_0=-i2n\pi T$, one obtains

$$T\sum_{k_0}\mathrm{Tr}_s[\gamma^\mu G_0^\pm(K_1)\,\gamma^\nu G_0^\pm(K_2)] = -\int_0^{1/T}d\tau\, e^{p_0\tau}\,\mathrm{Tr}_s[\gamma^\mu G_0^\pm(\tau,\mathbf{k}_1)\,\gamma^\nu\,\gamma_0\,G_0^\mp(\tau,\mathbf{k}_2)\,\gamma_0]. \tag{39}$$

One now inserts the expressions (32a),(32b), and integrates over τ. Putting everything together, one obtains for the gluon self-energy:

$$\Pi_0^{\mu\nu}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} \left\{ \mathcal{T}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \left[\left(\frac{n_1^0\,(1-n_2^0)}{p_0+\epsilon_1^0+\epsilon_2^0} - \frac{(1-n_1^0)\,n_2^0}{p_0-\epsilon_1^0-\epsilon_2^0} \right)(1-N_1^0-N_2^0) \right.\right.$$
$$\left. + \left(\frac{(1-n_1^0)\,(1-n_2^0)}{p_0-\epsilon_1^0+\epsilon_2^0} - \frac{n_1^0\,n_2^0}{p_0+\epsilon_1^0-\epsilon_2^0} \right)(N_1^0-N_2^0) \right] + \mathcal{T}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2)$$
$$\left. \times \left[\left(\frac{(1-n_1^0)\,n_2^0}{p_0+\epsilon_1^0+\epsilon_2^0} - \frac{n_1^0\,(1-n_2^0)}{p_0-\epsilon_1^0-\epsilon_2^0} \right)(1-N_1^0-N_2^0) + \left(\frac{n_1^0\,n_2^0}{p_0-\epsilon_1^0+\epsilon_2^0} - \frac{(1-n_1^0)\,(1-n_2^0)}{p_0+\epsilon_1^0-\epsilon_2^0} \right)(N_1^0-N_2^0) \right] \right\}. \tag{40}$$

Here,

$$\mathcal{T}_\pm^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \equiv \mathrm{Tr}_s(\gamma_0\,\gamma^\mu\,\Lambda_{\mathbf{k}_1}^{\pm e_1}\,\gamma_0\,\gamma^\nu\,\Lambda_{\mathbf{k}_2}^{\pm e_2}), \tag{41}$$

and I introduced the somewhat compact notation

$$\epsilon_i^0 \equiv \epsilon_{\mathbf{k}_i 0}^{e_i}, \quad n_i^0 \equiv n_{\mathbf{k}_i 0}^{e_i}, \quad N_i^0 \equiv N(\epsilon_i^0). \tag{42}$$

An (appropriately generalized) expression of the form (40) will also appear in Sec. IV, when the self-energy is computed in the superconducting phase. In the normal phase, however, one can use $n_i^0 \equiv \theta(\mu - e_i k_i)$ to show that

$$n_i^0\,(1-N_i^0) = n_i^0\,\{\theta(e_i)\,N_F^+(k_i) + \theta(-e_i)\,[1-N_F^-(k_i)]\}, \tag{43a}$$

$$(1-n_i^0)\,N_i^0 = (1-n_i^0)\,\{\theta(e_i)\,N_F^+(k_i) + \theta(-e_i)\,[1-N_F^-(k_i)]\} \tag{43b}$$

$$(1-n_i^0)\,(1-N_i^0) = (1-n_i^0)\,\{\theta(e_i)\,[1-N_F^+(k_i)] + \theta(-e_i)\,N_F^-(k_i)\}, \tag{43c}$$

$$n_i^0\,N_i^0 = n_i^0\,\{\theta(e_i)\,[1-N_F^+(k_i)] + \theta(-e_i)\,N_F^-(k_i)\}. \tag{43d}$$

Equation (40) then simplifies to

$$\Pi_0^{\mu\nu}(P) = \frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} \left[\frac{\mathcal{T}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2)}{p_0 - e_1 k_1 + e_2 k_2} - \frac{\mathcal{T}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2)}{p_0 + e_1 k_1 - e_2 k_2} \right]$$
$$\times \{\theta(e_1)[1-N_F^+(k_1)] + \theta(-e_1)\,N_F^-(k_1) - \theta(e_2)[1-N_F^+(k_2)] - \theta(-e_2)\,N_F^-(k_2)\}. \tag{44}$$

E. Trace over spinor space

The traces (41) are best computed for temporal and spatial components separately,

$$\mathcal{T}_\pm^{00} = 1 + e_1 e_2\,\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2, \tag{45a}$$

$$\mathcal{T}_\pm^{0i} = \mathcal{T}_\pm^{i0} = \pm e_1\,\hat{k}_1^i \pm e_2\,\hat{k}_2^i, \quad i = x,y,z, \tag{45b}$$

$$\mathcal{T}_\pm^{ij} = \delta^{ij}\,(1 - e_1 e_2\,\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2) + e_1 e_2\,(\hat{k}_1^i\,\hat{k}_2^j + \hat{k}_1^j\,\hat{k}_2^i),$$
$$i,j = x,y,z. \tag{45c}$$

Equation (40), or Eq. (44), together with Eqs. (45a)–(45c), completes the computation of the quark contribution to the gluon self-energy to one-loop order in the normal phase. At temperatures $T \ll \mu$, this is the dominant contribution to the gluon self-energy. In the following, I study the so-called hard-dense-loop (HDL) limit.

F. The HDL limit

To derive the HDL limit, it is advantageous to shift the integration over 3-momentum in Eqs. (40) or (44), $\mathbf{k} \to \mathbf{k} + \mathbf{p}/2$, such that $\mathbf{k}_1 = \mathbf{k} + \mathbf{p}/2$ and $\mathbf{k}_2 = \mathbf{k} - \mathbf{p}/2$. The HDL limit is obtained by taking p_0, p to be of order $g\mu$ (''soft''), while k is of order μ (''hard'') [15]. As the gluon self-energy (40) is already proportional to g^2, it is permissible to compute the integral in Eq. (40) to order $O(p^0)$. However, since some of the energy denominators are of order $O(p)$, one has to keep terms up to order $O(p)$ in the numerators, too. For the traces (45a)–(45c) one then obtains

$$\mathcal{T}_{\pm}^{00} \simeq 1 + e_1 e_2 + O\left(\frac{p^2}{k^2}\right), \tag{46a}$$

$$\mathcal{T}_{\pm}^{0i} = \mathcal{T}_{\pm}^{i0} \simeq \pm (e_1 + e_2) \hat{k}^i \pm (e_1 - e_2)(\delta^{ij} - \hat{k}^i \hat{k}^j) \frac{p^j}{2k} + O\left(\frac{p^2}{k^2}\right), \tag{46b}$$

$$\mathcal{T}_{\pm}^{ij} \simeq \delta^{ij}(1 - e_1 e_2) + 2\, e_1 e_2\, \hat{k}^i \hat{k}^j + O\left(\frac{p^2}{k^2}\right). \tag{46c}$$

In the following, the temporal and spatial components of the gluon self-energy are evaluated separately.

(i) $\mu = \nu = 0$. In the HDL limit, Eq. (46a) shows that only particle-particle $(e_1 = e_2 = +1)$, or antiparticle-antiparticle $(e_1 = e_2 = -1)$ excitations contribute to the electric components of the gluon self-energy. In this case, only the difference $k_1 - k_2$ occurs in the energy denominators in Eq. (44), which, in the HDL limit, is

$$k_1 - k_2 \simeq \mathbf{p} \cdot \hat{\mathbf{k}}. \tag{47}$$

In the numerators, the difference of the thermal occupation numbers is

$$N_F^{\pm}(k_1) - N_F^{\pm}(k_2) \simeq \mathbf{p} \cdot \hat{\mathbf{k}}\, \frac{dN_F^{\pm}(k)}{dk}. \tag{48}$$

Equation (44) with Eq. (46a) then yields

$$\Pi_0{}^{00}(P) \simeq g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(1 - \frac{p_0}{p_0 + \mathbf{p} \cdot \hat{\mathbf{k}}}\right) \times \left[\frac{dN_F^+(k)}{dk} + \frac{dN_F^-(k)}{dk}\right]. \tag{49}$$

With some effort, one can also do the integration over k exactly for *all* temperatures and chemical potentials [15]. In this case, the final answer encompasses not only the hard-dense-loop limit, but also the ''hard-thermal-loop'' (HTL) limit. That much effort is, however, not necessary in the present case. For superconductivity, one is interested in temperatures of the order of the zero-temperature gap, $T \sim \phi_0 \sim \mu \exp(-c_{\mathrm{QCD}}/g) \ll \mu$. On this basis it was argued above that contributions from the gluon and ghost loops to the gluon self-energy can be neglected, as they are $\sim g^2 T^2$, while the dominant contribution from the quark loop is $\sim g^2 \mu^2$.

In essence this means that effects from nonzero temperature can be neglected to leading order. Consequently,

$$\frac{dN_F^+(k)}{dk} \simeq \frac{d\, \theta(\mu - k)}{dk} = -\delta(k - \mu),$$

$$\frac{dN_F^-(k)}{dk} \simeq \frac{d\, \theta(k + \mu)}{dk} = 0. \tag{50}$$

From the physical point of view this is an important relation: only quark excitations *at* the Fermi surface contribute to the gluon self-energy.

With these approximations one obtains the well-known result [15]

$$\Pi_0{}^{00}(P) \simeq -3\, m_g^2 \int \frac{d\Omega}{4\pi} \left(1 - \frac{p_0}{p_0 + \mathbf{p} \cdot \hat{\mathbf{k}}}\right), \tag{51}$$

where $d\Omega$ is the integration over solid angle and

$$m_g^2 \equiv g^2 \frac{N_f}{6\pi^2} \mu^2 \tag{52}$$

is the gluon mass at $T = 0$. Equation (51) remains valid in the HTL limit, when Eq. (52) is properly generalized to nonzero temperature [15].

In the static limit, $p_0 = 0$, the dependence on $\mathbf{p}$ vanishes, and one simply has

$$\Pi_0{}^{00}(0) \simeq -3\, m_g^2, \tag{53}$$

the usual result for Debye screening.

(ii) $\mu = 0, \nu = i$. For Π_0^{0i}, one concludes from Eqs. (44) and (46b) that particle-antiparticle excitations are at least of order $O(p^2)$, i.e., to leading order in the HDL limit only particle-particle or antiparticle-antiparticle excitations contribute to the gluon self-energy. Then, with Eqs. (47) and (48) one obtains

$$\Pi_0{}^{0i}(P) \simeq g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{p_0\, \hat{k}^i}{p_0 + \mathbf{p} \cdot \hat{\mathbf{k}}} \left[\frac{dN_F^+(k)}{dk} + \frac{dN_F^-(k)}{dk}\right]. \tag{54}$$

For the temperatures of interest, one can again make the approximation (50) to obtain

$$\Pi_0{}^{0i}(P) \simeq -3\, m_g^2 \int \frac{d\Omega}{4\pi} \frac{p_0\, \hat{k}^i}{p_0 + \mathbf{p} \cdot \hat{\mathbf{k}}}, \tag{55}$$

which coincides with [15].

In the static limit,

$$\Pi_0{}^{0i}(0) \simeq 0. \tag{56}$$

(iii) $\mu = i$, $\nu = j$. For $\Pi_0{}^{ij}$, Eq. (46c) shows that not only particle-particle $(e_1 = e_2 = +1)$ and antiparticle-antiparticle $(e_1 = e_2 = -1)$ excitations contribute, as for the other components of $\Pi_0{}^{\mu\nu}$, but also particle-antiparticle $(e_1 = -e_2 = \pm 1)$ excitations. In the former, one encounters again the difference of momenta (47) and thermal occupation numbers (48). In the latter, however, the sum of momenta and thermal occupation numbers occurs. To leading order in p,

$$k_1 + k_2 \simeq 2\, k, \quad N_F^{\pm}(k_1) + N_F^{\pm}(k_2) \simeq 2\, N_F^{\pm}(k). \tag{57}$$

Then,

$$\Pi_0{}^{ij}(P) \simeq g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Bigg\{ \hat{k}^i \hat{k}^j \left(1 - \frac{p_0}{p_0 + \mathbf{p}\cdot\hat{\mathbf{k}}} \right) \times \left[\frac{dN_F^+(k)}{dk} + \frac{dN_F^-(k)}{dk} \right] - (\delta^{ij} - \hat{k}^i \hat{k}^j) \frac{1}{k} [1 - N_F^+(k) - N_F^-(k)] \Bigg\}. \tag{58}$$

The 1 in the last term is an ultraviolet-divergent vacuum contribution and has to be removed by renormalization. The angular integration can be performed for the parts which do not depend on $\mathbf{p}$, $\int (d\Omega/4\pi)\, \hat{k}^i \hat{k}^j \equiv \delta^{ij}/3$. One then realizes after an integration by parts that the $\mathbf{p}$-independent part of the first line in Eq. (58) cancels the second line,

$$\Pi_0{}^{ij}(P) \simeq -g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hat{k}^i \hat{k}^j \frac{p_0}{p_0 + \mathbf{p}\cdot\hat{\mathbf{k}}} \times \left[\frac{dN_F^+(k)}{dk} + \frac{dN_F^-(k)}{dk} \right]. \tag{59}$$

With the gluon mass (52) this can be written in the form

$$\Pi_0{}^{ij}(P) \simeq 3\, m_g^2 \int \frac{d\Omega}{4\pi} \hat{k}^i \hat{k}^j \frac{p_0}{p_0 + \mathbf{p}\cdot\hat{\mathbf{k}}}, \tag{60}$$

which is the standard result [15]. Static magnetic gluons are not screened,

$$\Pi_0{}^{ij}(0) \simeq 0. \tag{61}$$

IV. THE GLUON SELF-ENERGY IN THE SUPERCONDUCTING PHASE

In the superconducting phase, the ground state is a condensate of quark Cooper pairs, $\langle \overline{\psi}_C\, \psi \rangle \neq 0$. As was shown in [9], in mean-field approximation the quark propagator (19) becomes

$$\mathcal{S}(K) = \begin{pmatrix} G^+(K) & \Xi^-(K) \\ \Xi^+(K) & G^-(K) \end{pmatrix}, \tag{62}$$

where the quasiparticle and charge conjugate quasiparticle propagators are

$$G^\pm \equiv ([G_0^\pm]^{-1} - \Sigma^\pm)^{-1}, \quad \Sigma^\pm \equiv \Phi^\mp G_0^\mp \Phi^\pm. \tag{63}$$

$\Sigma^\pm$ is the quark self-energy generated by exchanging particles or charge conjugate particles with the condensate. For Σ^+, a particle annihilates with an antiparticle in the condensate $\Phi^+ \sim \langle \psi_C\, \overline{\psi} \rangle$, and a charge conjugate particle is created. This charge conjugate particle propagates via G_0^-, until it annihilates in the condensate $\Phi^- \sim \langle \psi\, \overline{\psi}_C \rangle$ with a charge conjugate antiparticle, whereby a particle is created [6]. The meaning of Σ^- can be explained analogously, except that the roles of particles and charge conjugate particles are interchanged.

The off-diagonal components of the quark propagator (62) are

$$\Xi^\pm \equiv -G_0^\mp \Phi^\pm G^\pm. \tag{64}$$

The physical interpretation is that particles (charge conjugate particles) annihilate with an antiparticle (a charge conjugate antiparticle) in the condensate, upon which a charge conjugate particle (a particle) is created.

In mean-field approximation, the condensate Φ^+ obeys the gap equation [5,6,9]

$$\Phi^+(K) \equiv -g^2 \frac{T}{V} \sum_Q \Delta^{ab}_{\mu\nu}(K-Q)\, \overline{\Gamma}^\mu_a\, \Xi^+(Q)\, \Gamma^\nu_b, \tag{65}$$

and Φ^- can be obtained from

$$\Phi^-(K) \equiv \gamma_0 [\Phi^+(K)]^\dagger \gamma_0 . \tag{66}$$

The solution of the gap equation (65) has been extensively discussed in [6].

The gluon self-energy (20) becomes

$$\Pi^{\mu\nu}_{ab}(P) = \frac{1}{2} g^2 \frac{T}{V} \sum_K \mathrm{Tr}_{s,c,f,NG} [\hat{\Gamma}^\mu_a\, \mathcal{S}(K)\, \hat{\Gamma}^\nu_b\, \mathcal{S}(K-P)]. \tag{67}$$

As in the normal phase, this expression is computed in several steps.

A. Trace over Nambu-Gor'kov space

The trace over the 2-dimensional Nambu–Gor'kov space is readily performed with Eqs. (11) and (62),

$$\begin{aligned} \Pi^{\mu\nu}_{ab}(P) = \frac{1}{2} g^2 \frac{T}{V} \sum_K \mathrm{Tr}_{s,c,f} [& \Gamma^\mu_a\, G^+(K)\, \Gamma^\nu_b\, G^+(K-P) \\ & + \overline{\Gamma}^\mu_a\, G^-(K)\, \overline{\Gamma}^\nu_b\, G^-(K-P) \\ & + \Gamma^\mu_a\, \Xi^-(K)\, \overline{\Gamma}^\nu_b\, \Xi^+(K-P) \\ & + \overline{\Gamma}^\mu_a\, \Xi^+(K)\, \Gamma^\nu_b\, \Xi^-(K-P)]. \end{aligned} \tag{68}$$

When the temperature approaches the critical temperature, $T \to T_c$, the condensate melts, $\Phi^\pm \to 0$, i.e., $\Xi^\pm \to 0$ and $G^\pm \to G_0^\pm$, and the gluon self-energy assumes the form of the normal phase, $\Pi^{\mu\nu}_{ab} \to \Pi_0{}^{\mu\nu}_{ab}$, which was discussed in the previous Sec. III.

B. Trace over flavor space

For a condensate with total spin $J=0$ and $N_f=2$, the condensate is totally antisymmetric in flavor space [7],

$$(\Phi^\pm)_{fg} \equiv \pm \epsilon_{fg}\, \Phi^\pm, \tag{69}$$

where use has been made of $\epsilon_{fg}^T = \epsilon_{gf} = -\epsilon_{fg}$. Consequently, since the free quark propagator is diagonal in flavor space, the quark self-energy is also diagonal in flavor space,

$$(\Sigma^\pm)_{fg} = (\Phi^\mp)_{fh}\,(G_0^\mp)_{hm}\,(\Phi^\pm)_{mg} = \epsilon_{hf}\,\epsilon_{hg}\,\Phi^\mp\,G_0^\mp\,\Phi^\pm = \delta_{fg}\,\Sigma^\pm. \tag{70}$$

Then, also the quasiparticle propagator is diagonal in flavor space,

$$(G^\pm)_{fg} = \delta_{fg}\,G^\pm. \tag{71}$$

On the other hand, the off-diagonal components of $\mathcal{S}$ are antisymmetric in flavor space,

$$(\Xi^\pm)_{fg} = -(G_0^\mp)_{fh}\,(\Phi^\pm)_{hm}\,(G^\pm)_{mg} = \pm\,\epsilon_{fg}\,\Xi^\pm. \tag{72}$$

As the vertices Γ_a^μ and $\overline{\Gamma}_a^\mu$ are flavor-blind (proportional to the unit matrix in flavor space), the trace over flavor space in Eq. (68) results in

$$\begin{aligned}\Pi_{ab}^{\mu\nu}(P) = \frac{1}{2} g^2 N_f\ \frac{T}{V}\sum_K\ \mathrm{Tr}_{s,c}[&\Gamma_a^\mu\,G^+(K)\,\Gamma_b^\nu\,G^+(K-P)\\ &+\overline{\Gamma}_a^\mu\,G^-(K)\,\overline{\Gamma}_b^\nu\,G^-(K-P)\\ &+\Gamma_a^\mu\,\Xi^-(K)\,\overline{\Gamma}_b^\nu\,\Xi^+(K-P)\\ &+\overline{\Gamma}_a^\mu\,\Xi^+(K)\,\Gamma_b^\nu\,\Xi^-(K-P)],\end{aligned} \tag{73}$$

where, of course, $N_f = 2$.

C. Trace over color space

An $N_f = 2$, $J = 0$ condensate is also totally antisymmetric in color space [7],

$$(\Phi^\pm)_{ij} \equiv \pm\,\epsilon_{ij3}\,\Phi^\pm, \tag{74}$$

where use has been made of $\epsilon_{ij3}^T = \epsilon_{ji3} = -\epsilon_{ij3}$, and where a global color rotation has been performed to orient the condensate into the (anti-)3-direction in color space. (The notation is again sloppy: the ‘‘3’’ is actually not a triplet, but an anti-triplet index.)

The free quark propagator is diagonal in color space, so that one computes for the quark self-energy

$$(\Sigma^\pm)_{ij} = (\Phi^\mp)_{ik}\,(G_0^\mp)_{kl}\,(\Phi^\pm)_{lj} = \epsilon_{ki3}\,\epsilon_{kj3}\,\Phi^\mp\,G_0^\mp\,\Phi^\pm = (\delta_{ij} - \delta_{i3}\,\delta_{j3})\,\Sigma^\pm. \tag{75}$$

This result is physically easy to interpret, remembering the above discussion of how the quark self-energy arises. Quarks with color 3 do not condense, consequently there is no antiquark in the condensate which a color-3 quark could annihilate with. Thus, color-3 quarks do not attain a self-energy [6].

The color structure of the quasiparticle propagator is therefore

$$(G^\pm)_{ij} = (\delta_{ij} - \delta_{i3}\,\delta_{j3})\,G^\pm + \delta_{i3}\,\delta_{j3}\,G_0^\pm. \tag{76}$$

For the off-diagonal components of $\mathcal{S}$ one then finds

$$(\Xi^\pm)_{ij} = -(G_0^\mp)_{ik}\,(\Phi^\pm)_{kl}\,(G^\pm)_{lj} = \pm\,\epsilon_{ij3}\,\Xi^\pm. \tag{77}$$

One now computes the trace over color space with the explicit form of the Gell-Mann matrices. After a somewhat tedious, but straightforward calculation one obtains for $a = b = 1,2,3$:

$$\begin{aligned}\Pi_{11}^{\mu\nu}(P) = \frac{1}{4} g^2 N_f\ \frac{T}{V}\sum_K\ \mathrm{Tr}_s[&\gamma^\mu\,G^+(K)\,\gamma^\nu\,G^+(K-P)\\ &+\gamma^\mu\,G^-(K)\,\gamma^\nu\,G^-(K-P)\\ &+\gamma^\mu\,\Xi^-(K)\,\gamma^\nu\,\Xi^+(K-P)\\ &+\gamma^\mu\,\Xi^+(K)\,\gamma^\nu\,\Xi^-(K-P)],\end{aligned} \tag{78a}$$

for $a = b = 4,5,6,7$:

$$\begin{aligned}\Pi_{44}^{\mu\nu}(P) = \frac{1}{8} g^2 N_f\ \frac{T}{V}\sum_K\ \mathrm{Tr}_s[&\gamma^\mu\,G_0^+(K)\,\gamma^\nu\,G^+(K-P)\\ &+\gamma^\mu\,G^+(K)\,\gamma^\nu\,G_0^+(K-P)\\ &+\gamma^\mu\,G_0^-(K)\,\gamma^\nu\,G^-(K-P)\\ &+\gamma^\mu\,G^-(K)\,\gamma^\nu\,G_0^-(K-P)],\end{aligned} \tag{78b}$$

and for $a = b = 8$:

$$\Pi_{88}^{\mu\nu}(P) = \frac{2}{3}\,\Pi_0^{\mu\nu}(P) + \frac{1}{3}\,\tilde{\Pi}^{\mu\nu}(P),$$

$$\begin{aligned}\tilde{\Pi}^{\mu\nu}(P) = \frac{1}{4} g^2 N_f\ \frac{T}{V}\sum_K\ \mathrm{Tr}_s[&\gamma^\mu\,G^+(K)\,\gamma^\nu\,G^+(K-P)\\ &+\gamma^\mu\,G^-(K)\,\gamma^\nu\,G^-(K-P)\\ &-\gamma^\mu\,\Xi^-(K)\,\gamma^\nu\,\Xi^+(K-P)\\ &-\gamma^\mu\,\Xi^+(K)\,\gamma^\nu\,\Xi^-(K-P)],\end{aligned} \tag{78c}$$

where $\Pi_0^{\mu\nu}$ is the gluon self-energy in the normal phase, Eq. (27b).

Apart from the diagonal elements (78a)–(78c), after performing the color-trace one also finds the off-diagonal elements

$$\Pi_{45}^{\mu\nu}(P) = -\Pi_{54}^{\mu\nu}(P) = \Pi_{67}^{\mu\nu}(P) = -\Pi_{76}^{\mu\nu}(P) \equiv i\,\hat{\Pi}^{\mu\nu}(P),$$

$$\hat{\Pi}^{\mu\nu}(P) \equiv \frac{1}{8} g^2 N_f \, \frac{T}{V} \sum_K \, \mathrm{Tr}_s[\gamma^\mu G_0^+(K)\,\gamma^\nu G^+(K-P) - \gamma^\mu G^+(K)\,\gamma^\nu G_0^+(K-P) - \gamma^\mu G_0^-(K)\,\gamma^\nu G^-(K-P) + \gamma^\mu G^-(K)\,\gamma^\nu G_0^-(K-P)]. \tag{78d}$$

The occurrence of these off-diagonal elements bears no special physical meaning. It simply indicates that the inverse gluon propagator Δ^{-1} is not diagonal in the original basis of adjoint colors. For instance, in the (45)-subspace of adjoint colors Δ^{-1} has the form

$$\begin{pmatrix} \Delta_0^{-1} + \Pi_{44} & i\,\hat{\Pi} \\ -i\,\hat{\Pi} & \Delta_0^{-1} + \Pi_{44} \end{pmatrix}. \tag{79}$$

This Hermitian matrix is easily diagonalized by the unitary matrix

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{80}$$

In the new (diagonal) basis of adjoint colors,

$$\begin{pmatrix} \Delta_0^{-1} + \Pi_{44} + \hat{\Pi} & 0 \\ 0 & \Delta_0^{-1} + \Pi_{44} - \hat{\Pi} \end{pmatrix}. \tag{81}$$

Similar arguments hold for the (67)-subspace. Therefore, rotating into the new (diagonal) basis,

$$\Pi_{44} + \hat{\Pi} = \Pi_{66} + \hat{\Pi} \rightarrow \Pi_{44} = \Pi_{66},$$

$$\Pi_{44} - \hat{\Pi} = \Pi_{66} - \hat{\Pi} \rightarrow \Pi_{55} = \Pi_{77}. \tag{82}$$

In the following, only these diagonal gluon self energies will be considered. They read explicitly

$$\Pi_{44}^{\mu\nu}(P) = \frac{1}{4} g^2 N_f \, \frac{T}{V} \sum_K \, \mathrm{Tr}_s[\gamma^\mu G_0^+(K)\,\gamma^\nu G^+(K-P) + \gamma^\mu G^-(K)\,\gamma^\nu G_0^-(K-P)], \tag{83a}$$

$$\Pi_{55}^{\mu\nu}(P) = \frac{1}{4} g^2 N_f \, \frac{T}{V} \sum_K \, \mathrm{Tr}_s[\gamma^\mu G^+(K)\,\gamma^\nu G_0^+(K-P) + \gamma^\mu G_0^-(K)\,\gamma^\nu G^-(K-P)]. \tag{83b}$$

Remembering the explicit form of the Gell-Mann matrices, the results (78a),(78c), and (83a),(83b) are simple to interpret. Gluons of adjoint colors 1, 2, and 3 see only quarks in the condensate, with fundamental colors 1 and 2. Their self-energy has therefore contributions from the diagonal ($G^\pm$), as well as the off-diagonal ($\Xi^\pm$) components of the quark propagator (62).

Gluons of colors 4 and 5 ''see'' the uncondensed quark with fundamental color 3, but also the condensed quarks of color 1. Analogously, gluons of colors 6 and 7 see the uncondensed quark and the condensed quark of color 2. Therefore, the fermion loop in the self-energy contains one free propagator $G_0^\pm$, corresponding to the uncondensed quark, and one quasiparticle (charge conjugate quasiparticle) propagator $G^\pm$, corresponding to the quark in the condensate. As there is no way to annihilate a color-3 quark in the condensate, there is no contribution from the off-diagonal components of Eq. (62).

Finally, gluons of color 8 see the condensed quarks of colors 1 and 2, but also the uncondensed color-3 quark. The contribution to the gluon self-energy from the latter is equal to that in the normal phase, ${\Pi_0}^{\mu\nu}$, the factor 2/3 comes from the (33)-element of T^8. Apart from the prefactor 1/3, the contribution from the condensed quarks, $\tilde{\Pi}^{\mu\nu}$, looks similar to $\Pi_{11}^{\mu\nu}$, except that the sign of the last two terms is different. As will be seen below, this difference is important to keep gluons of colors 1, 2, and 3 massless, while the eighth gluon becomes massive. Note that, for QED, $\overline{\Gamma}_a^\mu \rightarrow \overline{\Gamma}^\mu = -\gamma^\mu$, $g \rightarrow e$. Thus, for $N_f = 2$, the contribution from the condensed quarks to the self-energy of gluons of color 8, $\tilde{\Pi}^{\mu\nu}$, is exactly g^2/e^2 of what one expects for the photon self-energy in an ordinary superconductor.

D. Mixed representations for the quark propagators

For $m = 0$, the quasiparticle propagator can be written in terms of chirality and energy projectors [6,9],

$$G^\pm(K) = \sum_{h=r,l} \sum_{e=\pm} \frac{\mathcal{P}_{\pm h}\,\Lambda_{\mathbf{k}}^{\pm e}}{k_0^2 - [\epsilon_{\mathbf{k}}^e(\phi_h^e)]^2} [G_0^\mp(K)]^{-1}, \tag{84}$$

where $\mathcal{P}_{r,l} = (1 \pm \gamma_5)/2$ are chirality projectors (the notation $-h$ stands for l, if $h = r$, and r, if $h = l$). The quasiparticle energies are

$$\epsilon_{\mathbf{k}}^e(\phi_h^e) \equiv \sqrt{(\mu - ek)^2 + |\phi_h^e|^2}, \tag{85}$$

where ϕ_h^e is the gap function for pairing of quarks ($e = +1$) or antiquarks ($e = -1$) with chirality h.

An analysis of the gap functions in mean-field approximation shows [6] that left- and right-handed gap functions differ only by a complex phase factor,

$$\phi_r^e = \phi^e \exp(i\,\theta^e), \quad \phi_l^e = -\phi^e \exp(-i\,\theta^e), \tag{86}$$

with $\phi^e \in \mathbf{R}$. Moreover, the phase factor is independent of the energy projection, $\theta^+ = \theta^- \equiv \theta$. Condensation fixes the value of θ, and breaks the $U_A(1)$ symmetry (which is effectively restored at high densities) spontaneously. If $\theta = 0$ or $\pi/2$, condensation occurs in a spin-zero channel with good parity, $J^P = 0^+$ or $J^P = 0^-$, respectively. For $\theta \neq 0$, there is always a $J^P=0^-$ admixture, thus condensation breaks also parity [7,18]. For the sake of simplicity, in the following we only consider $\theta=0$.

From Eq. (86), $|\phi_r^e|\equiv|\phi_l^e|\equiv\phi^e$, and the sum over chiralities in Eq. (84) is superfluous. Writing the inverse free propagator as $[G_0^{\mp}(K)]^{-1}=[k_0\mp(\mu-ek)\mp 2\, ek\,\Lambda_{\mathbf{k}}^{\mp e}]\gamma_0$, Eq. (84) can be brought in the form

$$G^{\pm}(K)=\sum_{e=\pm}\frac{k_0\mp(\mu-ek)}{k_0^2-[\epsilon_{\mathbf{k}}^e]^2}\Lambda_{\mathbf{k}}^{\pm e}\,\gamma_0, \tag{87}$$

which should be compared with Eq. (28). Obviously, all that has changed is that the free quark excitation energies (29) have been replaced by the quasiparticle excitation energies (85), $\epsilon_{\mathbf{k}0}^e\to\epsilon_{\mathbf{k}}^e\equiv\epsilon_{\mathbf{k}}^e(\phi^e)$.

After realizing this, by comparison with Eqs. (32a),(32b) one can immediately write down the mixed representation for the quasiparticle propagators,

$$\begin{aligned}G^{+}(\tau,\mathbf{k})=-\sum_{e=\pm}\Lambda_{\mathbf{k}}^e\,\gamma_0\{&(1-n_{\mathbf{k}}^e)\,[\theta(\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(-\epsilon_{\mathbf{k}}^e\tau)-n_{\mathbf{k}}^e[\theta(-\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(\epsilon_{\mathbf{k}}^e\tau)\},\end{aligned} \tag{88a}$$

$$\begin{aligned}G^{-}(\tau,\mathbf{k})=-\sum_{e=\pm}\gamma_0\,\Lambda_{\mathbf{k}}^e\{&n_{\mathbf{k}}^e[\theta(\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(-\epsilon_{\mathbf{k}}^e\tau)-(1-n_{\mathbf{k}}^e)\,[\theta(-\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(\epsilon_{\mathbf{k}}^e\tau)\}.\end{aligned} \tag{88b}$$

Here,

$$n_{\mathbf{k}}^e\equiv\frac{\epsilon_{\mathbf{k}}^e+\mu-ek}{2\,\epsilon_{\mathbf{k}}^e} \tag{89}$$

are the occupation numbers for quasiparticles ($e=+1$) or quasi-antiparticles ($e=-1$) at zero temperature [9]. Consequently, $1-n_{\mathbf{k}}^e$ are the occupation numbers for quasiparticle holes or quasi-antiparticle holes. Due to the presence of a gap ϕ^e in the quasiparticle excitation spectrum, these occupation numbers are no longer simple theta functions in momentum space, as in the noninteracting case; the theta functions become ''smeared'' over a range $\sim\phi^e$ around the Fermi surface (cf. Fig. 2 in [9]). The relations (34) and (35) are also satisfied by $G^{\pm}(\tau,\mathbf{k})$.

From a comparison of Eqs. (88a),(88b) and (32a), (32b), one can immediately deduce from Eq. (40) the result for the traces $\mathrm{Tr}_s[\gamma^\mu G^{\pm}(K)\,\gamma^\nu G^{\pm}(K-P)]$, $\mathrm{Tr}_s[\gamma^\mu G_0^{\pm}(K)\gamma^\nu G^{\pm}(K-P)]$, or $\mathrm{Tr}_s[\gamma^\mu G^{\pm}(K)\,\gamma^\nu G_0^{\pm}(K-P)]$. All one has to do is replace

$$\epsilon_i^0\to\epsilon_i\equiv\epsilon_{\mathbf{k}_i}^{e_i},\quad n_i^0\to n_i\equiv n_{\mathbf{k}_i}^{e_i},\quad N_i^0\to N_i\equiv N(\epsilon_i), \tag{90}$$

when a propagator $G^{\pm}$ occurs in place of $G_0^{\pm}$.

One also needs a mixed representation for the off-diagonal components of $\mathcal{S}(K)$. First, write $\Xi^{\pm}(K)$, Eq. (64), in terms of projectors,

$$\Xi^{+}(K)=-\sum_{h=r,l}\sum_{e=\pm}\frac{\phi_h^e(K)}{k_0^2-[\epsilon_{\mathbf{k}}^e]^2}\mathcal{P}_{-h}\,\Lambda_{\mathbf{k}}^{-e},$$

$$\Xi^{-}(K)=-\sum_{h=r,l}\sum_{e=\pm}\frac{[\phi_h^e(K)]^*}{k_0^2-[\epsilon_{\mathbf{k}}^e]^2}\mathcal{P}_h\,\Lambda_{\mathbf{k}}^{e}. \tag{91}$$

As in [6], assume that $\phi_h^e(k_0)$ has no poles or cuts in the complex k_0-plane and that $\phi_h^e(k_0)=\phi_h^e(-k_0)$. In this case, one obtains the mixed representations

$$\begin{aligned}\Xi^{+}(\tau,\mathbf{k})=\sum_{h=r,l}\sum_{e=\pm}\mathcal{P}_{-h}\,\Lambda_{\mathbf{k}}^{-e}\frac{\phi_h^e(\epsilon_{\mathbf{k}}^e,\mathbf{k})}{2\,\epsilon_{\mathbf{k}}^e}\{&[\theta(\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(-\epsilon_{\mathbf{k}}^e\tau)+[\theta(-\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(\epsilon_{\mathbf{k}}^e\tau)\},\end{aligned} \tag{92a}$$

$$\begin{aligned}\Xi^{-}(\tau,\mathbf{k})=\sum_{h=r,l}\sum_{e=\pm}\mathcal{P}_{h}\,\Lambda_{\mathbf{k}}^{e}\frac{[\phi_h^e(\epsilon_{\mathbf{k}}^e,\mathbf{k})]^*}{2\,\epsilon_{\mathbf{k}}^e}\{&[\theta(\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(-\epsilon_{\mathbf{k}}^e\tau)+[\theta(-\tau)-N(\epsilon_{\mathbf{k}}^e)]\\&\times\exp(\epsilon_{\mathbf{k}}^e\tau)\}.\end{aligned} \tag{92b}$$

Note that the energy in the gap functions ϕ_h^e is on the quasiparticle mass shell, $k_0\equiv\pm\epsilon_{\mathbf{k}}^e$.

The traces in Eqs. (78a)–(78d) involving $\Xi^{\pm}$ are now straightforwardly computed as

$$\begin{aligned}&T\sum_{k_0}\mathrm{Tr}_s[\gamma^\mu\,\Xi^{\mp}(K_1)\,\gamma^\nu\,\Xi^{\pm}(K_2)]\\&\quad=\sum_{e_1,e_2=\pm}\mathcal{U}_{\pm}^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2)\,\frac{\phi_1\,\phi_2}{4\,\epsilon_1\,\epsilon_2}\\&\qquad\times\Bigg[\left(\frac{1}{p_0+\epsilon_1+\epsilon_2}-\frac{1}{p_0-\epsilon_1-\epsilon_2}\right)(1-N_1-N_2)\\&\qquad-\left(\frac{1}{p_0-\epsilon_1+\epsilon_2}-\frac{1}{p_0+\epsilon_1-\epsilon_2}\right)(N_1-N_2)\Bigg],\end{aligned} \tag{93}$$

where $K_1\equiv K$, $K_2\equiv K-P$, as before, while

$$\phi_i\equiv\phi^{e_i}(\epsilon_i,\mathbf{k}_i), \tag{94}$$

and

$$\mathcal{U}_{\pm}^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2)\equiv\mathrm{Tr}_s[\gamma^\mu\,\Lambda_{\mathbf{k}_1}^{\pm e_1}\,\gamma^\nu\,\Lambda_{\mathbf{k}_2}^{\mp e_2}]. \tag{95}$$

On account of $\mathcal{P}_h\,\gamma^\mu=\gamma^\mu\,\mathcal{P}_{-h}$ and $\mathcal{P}_r\,\mathcal{P}_l=0$, the sum over chiralities h_1 and h_2 originating from the mixed representations (92a),(92b) could be performed trivially.

Putting everything together, the self-energy for gluons of color 1, 2, and 3 is

$$\Pi_{11}^{\mu\nu}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} \Bigg\{ \mathcal{T}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{n_1(1-n_2)}{p_0+\epsilon_1+\epsilon_2} - \frac{(1-n_1)\,n_2}{p_0-\epsilon_1-\epsilon_2} \right)(1-N_1-N_2) + \left(\frac{(1-n_1)(1-n_2)}{p_0-\epsilon_1+\epsilon_2} - \frac{n_1 n_2}{p_0+\epsilon_1-\epsilon_2} \right)(N_1-N_2) \Bigg] + \mathcal{T}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{(1-n_1)\,n_2}{p_0+\epsilon_1+\epsilon_2} - \frac{n_1(1-n_2)}{p_0-\epsilon_1-\epsilon_2} \right)(1-N_1-N_2) + \left(\frac{n_1 n_2}{p_0-\epsilon_1+\epsilon_2} - \frac{(1-n_1)(1-n_2)}{p_0+\epsilon_1-\epsilon_2} \right)(N_1-N_2) \Bigg] - [\mathcal{U}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) + \mathcal{U}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2)] \frac{\phi_1\,\phi_2}{4\,\epsilon_1\,\epsilon_2} \Bigg[\left(\frac{1}{p_0+\epsilon_1+\epsilon_2} - \frac{1}{p_0-\epsilon_1-\epsilon_2} \right)(1-N_1-N_2) - \left(\frac{1}{p_0-\epsilon_1+\epsilon_2} - \frac{1}{p_0+\epsilon_1-\epsilon_2} \right)(N_1-N_2) \Bigg] \Bigg\}, \tag{96a}$$

for gluon colors 4 and 6,

$$\Pi_{44}^{\mu\nu}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} \Bigg\{ \mathcal{T}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{n_1^0(1-n_2)}{p_0+\epsilon_1^0+\epsilon_2} - \frac{(1-n_1^0)\,n_2}{p_0-\epsilon_1^0-\epsilon_2} \right)(1-N_1^0-N_2) + \left(\frac{(1-n_1^0)(1-n_2)}{p_0-\epsilon_1^0+\epsilon_2} - \frac{n_1^0 n_2}{p_0+\epsilon_1^0-\epsilon_2} \right)(N_1^0-N_2) \Bigg] + \mathcal{T}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{(1-n_1)\,n_2^0}{p_0+\epsilon_1+\epsilon_2^0} - \frac{n_1(1-n_2^0)}{p_0-\epsilon_1-\epsilon_2^0} \right)(1-N_1-N_2^0) + \left(\frac{n_1 n_2^0}{p_0-\epsilon_1+\epsilon_2^0} - \frac{(1-n_1)(1-n_2^0)}{p_0+\epsilon_1-\epsilon_2^0} \right)(N_1-N_2^0) \Bigg] \Bigg\}, \tag{96b}$$

for gluon colors 5 and 7,

$$\Pi_{55}^{\mu\nu}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} \Bigg\{ \mathcal{T}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{n_1(1-n_2^0)}{p_0+\epsilon_1+\epsilon_2^0} - \frac{(1-n_1)\,n_2^0}{p_0-\epsilon_1-\epsilon_2^0} \right)(1-N_1-N_2^0) + \left(\frac{(1-n_1)(1-n_2^0)}{p_0-\epsilon_1+\epsilon_2^0} - \frac{n_1 n_2^0}{p_0+\epsilon_1-\epsilon_2^0} \right)(N_1-N_2^0) \Bigg] + \mathcal{T}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{(1-n_1^0)\,n_2}{p_0+\epsilon_1^0+\epsilon_2} - \frac{n_1^0(1-n_2)}{p_0-\epsilon_1^0-\epsilon_2} \right)(1-N_1^0-N_2) + \left(\frac{n_1^0 n_2}{p_0-\epsilon_1^0+\epsilon_2} - \frac{(1-n_1^0)(1-n_2)}{p_0+\epsilon_1^0-\epsilon_2} \right)(N_1^0-N_2) \Bigg] \Bigg\}, \tag{96c}$$

and for gluon color 8

$$\tilde{\Pi}^{\mu\nu}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} \Bigg\{ \mathcal{T}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{n_1(1-n_2)}{p_0+\epsilon_1+\epsilon_2} - \frac{(1-n_1)\,n_2}{p_0-\epsilon_1-\epsilon_2} \right)(1-N_1-N_2) + \left(\frac{(1-n_1)(1-n_2)}{p_0-\epsilon_1+\epsilon_2} - \frac{n_1 n_2}{p_0+\epsilon_1-\epsilon_2} \right)(N_1-N_2) \Bigg] + \mathcal{T}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) \times \Bigg[\left(\frac{(1-n_1)\,n_2}{p_0+\epsilon_1+\epsilon_2} - \frac{n_1(1-n_2)}{p_0-\epsilon_1-\epsilon_2} \right)(1-N_1-N_2) + \left(\frac{n_1 n_2}{p_0-\epsilon_1+\epsilon_2} - \frac{(1-n_1)(1-n_2)}{p_0+\epsilon_1-\epsilon_2} \right)(N_1-N_2) \Bigg] + [\mathcal{U}_+^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2) + \mathcal{U}_-^{\mu\nu}(\mathbf{k}_1,\mathbf{k}_2)] \frac{\phi_1\,\phi_2}{4\,\epsilon_1\,\epsilon_2} \Bigg[\left(\frac{1}{p_0+\epsilon_1+\epsilon_2} - \frac{1}{p_0-\epsilon_1-\epsilon_2} \right)(1-N_1-N_2) - \left(\frac{1}{p_0-\epsilon_1+\epsilon_2} - \frac{1}{p_0+\epsilon_1-\epsilon_2} \right)(N_1-N_2) \Bigg] \Bigg\}. \tag{96d}$$

E. Trace over spinor space

The traces $\mathcal{T}_\pm^{\mu\nu}$ have been computed in Sec. III E. What remains to be done is to compute $\mathcal{U}_\pm^{\mu\nu}$. One finds

$$\mathcal{U}_\pm^{00} = \mathcal{T}_\pm^{00}, \tag{97a}$$

$$\mathcal{U}_{\pm}^{0i} = -\mathcal{U}_{\pm}^{i0} = -\mathcal{T}_{\pm}^{0i}, \quad i = x,y,x, \tag{97b}$$

$$\mathcal{U}_{\pm}^{ij} = -\mathcal{T}_{\pm}^{ij}, \quad i,j = x,y,z. \tag{97c}$$

In the following, the results for the different components of the gluon self-energy in the superconducting phase will be collected.

F. Gluons of color 1, 2, and 3

(i) $\mu = \nu = 0$. Defining

$$\xi_i \equiv e_i k_i - \mu, \tag{98}$$

the self-energy of electric gluons of color 1, 2, and 3 is determined from Eqs. (45a)–(45c), (96a), and (97a) as

$$\Pi_{11}^{00}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} (1 + e_1 e_2 \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) \times \Bigg[\left(\frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - N_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} + \left(\frac{1}{p_0 - \epsilon_1 + \epsilon_2} - \frac{1}{p_0 + \epsilon_1 - \epsilon_2} \right) (N_1 - N_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \Bigg]. \tag{99a}$$

(ii) $\mu = 0, \nu = i$. For the $(0i)$-components of the self-energy of gluons with colors 1, 2, or 3 one obtains

$$\Pi_{11}^{0i}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} (e_1 \hat{k}_1^i + e_2 \hat{k}_2^i) \times \Bigg[\left(\frac{1}{p_0 + \epsilon_1 + \epsilon_2} + \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - N_2) \left(\frac{\xi_2}{2 \epsilon_2} - \frac{\xi_1}{2 \epsilon_1} \right) + \left(\frac{1}{p_0 - \epsilon_1 + \epsilon_2} + \frac{1}{p_0 + \epsilon_1 - \epsilon_2} \right) (N_1 - N_2) \left(\frac{\xi_1}{2 \epsilon_1} + \frac{\xi_2}{2 \epsilon_2} \right) \Bigg]. \tag{99b}$$

(iii) $\mu = i$, $\nu = j$. The self-energy of magnetic gluons of colors 1, 2, and 3 is

$$\Pi_{11}^{ij}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} [\delta^{ij}(1 - e_1 e_2 \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) + e_1 e_2 (\hat{k}_1^i \hat{k}_2^j + \hat{k}_1^j \hat{k}_2^i)] \times \Bigg[\left(\frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - N_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} + \left(\frac{1}{p_0 - \epsilon_1 + \epsilon_2} - \frac{1}{p_0 + \epsilon_1 - \epsilon_2} \right) (N_1 - N_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 - \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \Bigg]. \tag{99c}$$

G. Gluons of color 4 and 6

(i) $\mu = \nu = 0$. Using the symmetry of Eq. (96b) under $\mathbf{k}_1 \leftrightarrow -\mathbf{k}_2$, $e_1 \leftrightarrow e_2$, the self-energy of electric gluons of colors 4 and 6 can be written as

$$\Pi_{44}^{00}(P) = -\frac{1}{2} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1,e_2=\pm} (1 + e_1 e_2 \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) \times \Bigg[\left(\frac{n_1^0 (1 - n_2)}{p_0 + \epsilon_1^0 + \epsilon_2} - \frac{(1 - n_1^0) n_2}{p_0 - \epsilon_1^0 - \epsilon_2} \right) (1 - N_1^0 - N_2) + \left(\frac{(1 - n_1^0)(1 - n_2)}{p_0 - \epsilon_1^0 + \epsilon_2} - \frac{n_1^0 n_2}{p_0 + \epsilon_1^0 - \epsilon_2} \right) (N_1^0 - N_2) \Bigg]. \tag{100a}$$

(ii) $\mu = 0, \nu = i$. The same symmetry arguments lead to

$$\Pi^{0i}_{44}(P)=-\frac{1}{2}g^2 N_f\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{e_1,e_2=\pm}(e_1\hat{k}^i_1+e_2\hat{k}^i_2)$$
$$\times\left[\left(\frac{n^0_1(1-n_2)}{p_0+\epsilon^0_1+\epsilon_2}-\frac{(1-n^0_1)n_2}{p_0-\epsilon^0_1-\epsilon_2}\right)(1-N^0_1-N_2)+\left(\frac{(1-n^0_1)(1-n_2)}{p_0-\epsilon^0_1+\epsilon_2}-\frac{n^0_1 n_2}{p_0+\epsilon^0_1-\epsilon_2}\right)(N^0_1-N_2)\right]. \tag{100b}$$

(iii) $\mu=i$, $\nu=j$. For the self-energy of magnetic gluons of color 4 and 6 one obtains

$$\Pi^{ij}_{44}(P)=-\frac{1}{2}g^2 N_f\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{e_1,e_2=\pm}[\delta^{ij}(1-e_1e_2\,\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+e_1e_2(\hat{k}^i_1\hat{k}^j_2+\hat{k}^j_1\hat{k}^i_2)]$$
$$\times\left[\left(\frac{n^0_1(1-n_2)}{p_0+\epsilon^0_1+\epsilon_2}-\frac{(1-n^0_1)n_2}{p_0-\epsilon^0_1-\epsilon_2}\right)(1-N^0_1-N_2)+\left(\frac{(1-n^0_1)(1-n_2)}{p_0-\epsilon^0_1+\epsilon_2}-\frac{n^0_1 n_2}{p_0+\epsilon^0_1-\epsilon_2}\right)(N^0_1-N_2)\right]. \tag{100c}$$

H. Gluons of color 5 and 7

(i) $\mu=\nu=0$. Again using the symmetry of Eq. (96c) under $\mathbf{k}_1\leftrightarrow-\mathbf{k}_2$, $e_1\leftrightarrow e_2$, the self-energy of electric gluons of colors 5 and 7 can be written as

$$\Pi^{00}_{55}(P)=-\frac{1}{2}g^2 N_f\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{e_1,e_2=\pm}(1+e_1e_2\,\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)$$
$$\times\left[\left(\frac{(1-n^0_1)n_2}{p_0+\epsilon^0_1+\epsilon_2}-\frac{n^0_1(1-n_2)}{p_0-\epsilon^0_1-\epsilon_2}\right)(1-N^0_1-N_2)+\left(\frac{n^0_1 n_2}{p_0-\epsilon^0_1+\epsilon_2}-\frac{(1-n^0_1)(1-n_2)}{p_0+\epsilon^0_1-\epsilon_2}\right)(N^0_1-N_2)\right]. \tag{101a}$$

(ii) $\mu=0,\nu=i$. The $(0i)$-components are

$$\Pi^{0i}_{55}(P)=\frac{1}{2}g^2 N_f\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{e_1,e_2=\pm}(e_1\hat{k}^i_1+e_2\hat{k}^i_2)$$
$$\times\left[\left(\frac{(1-n^0_1)n_2}{p_0+\epsilon^0_1+\epsilon_2}-\frac{n^0_1(1-n_2)}{p_0-\epsilon^0_1-\epsilon_2}\right)(1-N^0_1-N_2)+\left(\frac{n^0_1 n_2}{p_0-\epsilon^0_1+\epsilon_2}-\frac{(1-n^0_1)(1-n_2)}{p_0+\epsilon^0_1-\epsilon_2}\right)(N^0_1-N_2)\right]. \tag{101b}$$

(iii) $\mu=i$, $\nu=j$. For the self-energy of magnetic gluons of color 5 and 7 one gets

$$\Pi^{ij}_{55}(P)=-\frac{1}{2}g^2 N_f\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{e_1,e_2=\pm}[\delta^{ij}(1-e_1e_2\,\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+e_1e_2(\hat{k}^i_1\hat{k}^j_2+\hat{k}^j_1\hat{k}^i_2)]$$
$$\times\left[\left(\frac{(1-n^0_1)n_2}{p_0+\epsilon^0_1+\epsilon_2}-\frac{n^0_1(1-n_2)}{p_0-\epsilon^0_1-\epsilon_2}\right)(1-N^0_1-N_2)+\left(\frac{n^0_1 n_2}{p_0-\epsilon^0_1+\epsilon_2}-\frac{(1-n^0_1)(1-n_2)}{p_0+\epsilon^0_1-\epsilon_2}\right)(N^0_1-N_2)\right]. \tag{101c}$$

I. Gluons of color 8

(i) $\mu=\nu=0$. For $\tilde{\Pi}^{00}$ one obtains

$$\tilde{\Pi}^{00}(P)=-\frac{1}{4}g^2 N_f\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{e_1,e_2=\pm}(1+e_1e_2\,\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)$$
$$\times\left[\left(\frac{1}{p_0+\epsilon_1+\epsilon_2}-\frac{1}{p_0-\epsilon_1-\epsilon_2}\right)(1-N_1-N_2)\frac{\epsilon_1\,\epsilon_2-\xi_1\,\xi_2+\phi_1\,\phi_2}{2\,\epsilon_1\,\epsilon_2}\right.$$
$$\left.+\left(\frac{1}{p_0-\epsilon_1+\epsilon_2}-\frac{1}{p_0+\epsilon_1-\epsilon_2}\right)(N_1-N_2)\frac{\epsilon_1\,\epsilon_2+\xi_1\,\xi_2-\phi_1\,\phi_2}{2\,\epsilon_1\,\epsilon_2}\right]. \tag{102a}$$

(ii) $\mu=0,\nu=i$. For $\tilde{\Pi}^{0i}$ one simply has

$$\tilde{\Pi}^{0i}(P) \equiv \Pi_{11}^{0i}(P). \tag{102b}$$

(iii) $\mu = i$, $\nu = j$. The magnetic components $\tilde{\Pi}^{ij}$ are

$$\tilde{\Pi}^{ij}(P) = -\frac{1}{4} g^2 N_f \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{e_1, e_2 = \pm} [\delta^{ij}(1 - e_1 e_2 \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) + e_1 e_2 (\hat{k}_1^i \hat{k}_2^j + \hat{k}_1^j \hat{k}_2^i)]$$
$$\times \Bigg[\left(\frac{1}{p_0 + \epsilon_1 + \epsilon_2} - \frac{1}{p_0 - \epsilon_1 - \epsilon_2} \right) (1 - N_1 - N_2) \frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2}$$
$$+ \left(\frac{1}{p_0 - \epsilon_1 + \epsilon_2} - \frac{1}{p_0 + \epsilon_1 - \epsilon_2} \right) (N_1 - N_2) \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \Bigg]. \tag{102c}$$

Equations (99a)–(102c) are the central result of this work. Starting from these equations, one can derive explicit expressions for the gluon self-energy in a two-flavor color superconductor for arbitrary p_0 and $\mathbf{p}$. As a first step, in the remainder of this work I compute the color-electric (Debye) screening mass, as well as the color-magnetic (Meissner) mass. These are obtained from the gluon self-energy in the static limit, $p_0 = 0$, for $p \to 0$. Then I compute the self-energy for $p_0 = 0$, but $p \gg \phi_0$.

V. DEBYE SCREENING AND MEISSNER EFFECT

In the following, I shall always assume that antiparticle gaps are small, $\phi^- \simeq 0$, and consequently that

$$\epsilon_{\mathbf{k}}^- \simeq \epsilon_{\mathbf{k}0}^-, \quad n_{\mathbf{k}}^- \simeq n_{\mathbf{k}0}^- \simeq 1, \quad N(\epsilon_{\mathbf{k}}^-) \simeq 0. \tag{103}$$

Therefore, thermal antiparticle occupation numbers and their derivatives will be neglected. As in the previous section, the different color sectors will be discussed separately.

A. Gluons with colors 1, 2, and 3

(i) $\mu = \nu = 0$. I show several calculational steps in greater detail to illustrate the main approximations used throughout the following. For $p_0 = 0$, $p \to 0$, $\mathbf{k}_2 \to \mathbf{k}_1 \equiv \mathbf{k}$, and only particle-particle $(e_1 = e_2 = +1)$, or antiparticle-antiparticle $(e_1 = e_2 = -1)$ excitations contribute in the sum over e_1 and e_2 in Eq. (99a). This is very similar to what happens in the HDL limit, cf. Sec. III F. Furthermore

$$\frac{\epsilon_1 \epsilon_2 - \xi_1 \xi_2 - \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \to 0, \quad \frac{\epsilon_1 \epsilon_2 + \xi_1 \xi_2 + \phi_1 \phi_2}{2 \epsilon_1 \epsilon_2} \to 1. \tag{104}$$

In the limit $\mathbf{k}_2 \to \mathbf{k}_1$, $(N_1 - N_2)/(\epsilon_1 - \epsilon_2) \to dN/d\epsilon$, and neglecting the variation of $N(\epsilon_{\mathbf{k}}^-)$, as discussed above, one obtains

$$\Pi_{11}^{00}(0) \simeq \frac{g^2 N_f}{2\pi^2} \int_0^\infty dk\, k^2 \frac{dN(\epsilon_{\mathbf{k}}^+)}{d\epsilon_{\mathbf{k}}^+}. \tag{105}$$

As the thermal occupation number varies appreciably only close to the Fermi surface, it is permissible to approximate $k^2 \simeq \mu^2$, and to restrict the k integration to the region $0 \leqslant k \leqslant 2\mu$. Introducing the variable

$$\xi \equiv k - \mu, \tag{106}$$

one obtains with Eq. (52)

$$\Pi_{11}^{00}(0) \simeq -3\, m_g^2 \int_0^\mu \frac{d\xi}{2T} \frac{1}{\cosh^2(\sqrt{\xi^2 + \phi^2}/2T)}. \tag{107}$$

Now change variables to $\zeta \equiv \xi/2T$, and remembering that $\mu \gg \phi \sim T$, send the upper limit of the integral to infinity,

$$\Pi_{11}^{00}(0) \simeq -3\, m_g^2 \int_0^\infty d\zeta \frac{1}{\cosh^2 \sqrt{\zeta^2 + (\phi/2T)^2}}. \tag{108}$$

This expression has two interesting limits. For $T \to 0$, the integrand becomes zero everywhere, and

$$T \to 0: \quad \Pi_{11}^{00}(0) \to 0. \tag{109}$$

At zero temperature, static, homogeneous electric fields of colors 1, 2, or 3, are *not screened*.

The other limit is when $T \to T_c$, and $\phi \to 0$. Then, as $\int_0^\infty d\zeta/\cosh^2\zeta \equiv 1$,

$$T \to T_c: \quad \Pi_{11}^{00}(0) \to -3\, m_g^2 \equiv \Pi_0{}^{00}(0). \tag{110}$$

As expected, $\Pi_{11}^{00}(0)$ approaches the value in the normal phase, Eq. (53).

The interpretation of this result is the following. From the explicit form of the Gell-Mann matrices it is clear that gluons of adjoint colors 1, 2, and 3 ''see'' only quarks with fundamental colors 1 and 2. However, at $T = 0$, all these quarks are bound in Cooper pairs to form a condensate of fundamental color (anti-)3, to which these gluons are ''blind.'' Hence, at $T = 0$ the color superconductor is transparent with respect to these color fields. There is nothing which could screen these fields, thus there is no Debye mass for the gluons of colors 1, 2, or 3. Of course, this holds only in the limit $p_0 = 0$, $p \to 0$, because only then are the gluons unable to resolve the individual quarks (with colors that can be ''seen'') inside a Cooper pair.

When T is nonzero, quasiparticles are thermally excited, and screening sets in. As T approaches T_c, the condensate melts completely, and all quarks with the right colors to screen gluon fields with colors 1, 2, and 3 are freed. Then, the gluon self-energy approaches its value in the normal phase.

(ii) $\mu=0$, $\nu=i$. From Eq. (99b) it is clear that

$$\Pi_{11}^{0i}(0,\mathbf{p})\equiv 0. \tag{111}$$

This is similar to the normal phase in the static limit, Eq. (56).

(iii) $\mu=i$, $\nu=j$. As in the HDL limit, the magnetic components of the gluon self-energy receive contributions not only from particle-particle and antiparticle-antiparticle, but also from particle-antiparticle excitations. With Eq. (103) and $\int d\Omega\, \hat{k}^i \hat{k}^j/(4\pi)=\delta^{ij}/3$, one obtains from Eq. (99c)

$$\Pi_{11}^{ij}(0)\simeq-\delta^{ij}\frac{g^2 N_f}{6\pi^2}\int_0^\infty dk\,k^2\left\{\frac{[\phi_\mathbf{k}^+]^2}{2[\epsilon_\mathbf{k}^+]^3}\tanh\left(\frac{\epsilon_\mathbf{k}^+}{2T}\right)-\frac{dN(\epsilon_\mathbf{k}^+)}{d\epsilon_\mathbf{k}^+}\frac{\xi^2}{[\epsilon_\mathbf{k}^+]^2}+\frac{4[1-N(\epsilon_\mathbf{k}^+)](1-n_\mathbf{k}^+)}{\epsilon_\mathbf{k}^+ + k+\mu}-\frac{4N(\epsilon_\mathbf{k}^+)\,n_\mathbf{k}^+}{\epsilon_\mathbf{k}^+ - k-\mu}-\frac{2}{k}\right\}, \tag{112}$$

where the last term was added to subtract the (UV-divergent) vacuum contribution, and where $\phi_\mathbf{k}^+\equiv\phi^+(\epsilon_\mathbf{k}^+,\mathbf{k})$.

At zero temperature, and after an integration by parts ($dn_\mathbf{k}^+/dk=-[\phi_\mathbf{k}^+]^2/2[\epsilon_\mathbf{k}^+]^3$),

$$\Pi_{11}^{ij}(0)\simeq\delta^{ij}\frac{g^2 N_f}{6\pi^2}\int_0^\infty dk\,k\,\frac{4\,\epsilon_\mathbf{k}^+}{\epsilon_\mathbf{k}^+ + k+\mu}n_\mathbf{k}^+(1-n_\mathbf{k}^+). \tag{113}$$

The term $n_\mathbf{k}^+(1-n_\mathbf{k}^+)$ is proportional to $[\phi_\mathbf{k}^+]^2$. The momentum dependence of the gap function is $\phi_\mathbf{k}^+=\phi_0\sin(\bar{g}\,x_\mathbf{k})$ [5,6], where $\bar{g}=g/(3\sqrt{2}\pi)$ and $x_\mathbf{k}\simeq\ln[2b\mu/(\epsilon_\mathbf{k}^+ + |\xi|)]$, with ξ defined in Eq. (106) and $b\equiv 256\,\pi^4[2/(N_f g^2)]^{5/2}$. The gap function peaks at the Fermi surface, and is small far away from the Fermi surface. Therefore, the region $k\geqslant 2\mu$ can be neglected.

In the remaining integral over the region $0\leqslant k\leqslant 2\mu$, take $k\simeq\mu$ in the slowly varying factor $k/(\epsilon_\mathbf{k}^+ + k+\mu)$, and change the integration variable to ξ:

$$\Pi_{11}^{ij}(0)\simeq\delta^{ij}\frac{g^2 N_f}{6\pi^2}\int_0^\mu\frac{d\xi}{\epsilon_\mathbf{k}^+}[\phi_\mathbf{k}^+]^2. \tag{114}$$

Inserting the solution of the gap equation (including the momentum dependence), and changing the integration variable to $x=\ln[2b\mu/(\epsilon_\mathbf{k}^+ + \xi)]$, this integral can be solved analytically. However, it turns out that this is unnecessary, if one only wants to know the parametric dependence on the gap and the QCD coupling constant in weak coupling, $g\ll 1$. One can simply neglect the momentum dependence of the gap function, and approximate $\phi_\mathbf{k}^+$ by its value at the Fermi surface, ϕ_0, to obtain

$$\Pi_{11}^{ij}(0)\simeq\delta^{ij}\,m_g^2\frac{\phi_0^2}{\mu^2}\ln\left(\frac{2\mu}{\phi_0}\right). \tag{115}$$

As $\phi_0\sim\mu\exp(-c_{\rm QCD}/g)$, Π_{11}^{ij} is formally of order $\sim g\,\phi_0^2$. To this order, I cannot exclude that there are cancellations from other terms I have neglected (for instance the antiparticle gaps). To leading order, the result (115) is therefore consistent with $\Pi_{11}^{ij}(0)\simeq 0$.

Finally, as $T\to T_c$, an integration by parts shows that the expression (112) approaches the HDL limit, Eq. (61).

B. Gluons with colors 4 and 6

(i) $\mu=\nu=0$. For $p_0=0$, $p\to 0$, and with the approximations (103), Eq. (100a) becomes

$$\Pi_{44}^{00}(0)\simeq-\frac{g^2 N_f}{2\pi^2}\int_0^\infty dk\,k^2\left\{\frac{1-n_\mathbf{k}^+}{\epsilon_\mathbf{k}^+ - \xi}[N_F^+(k)-N(\epsilon_\mathbf{k}^+)]+\frac{n_\mathbf{k}^+}{\epsilon_\mathbf{k}^+ + \xi}[1-N_F^+(k)-N(\epsilon_\mathbf{k}^+)]\right\}. \tag{116}$$

At $T=0$, and restricting the k integration to the range $0\leqslant k\leqslant 2\mu$ (as before, the momentum dependence of the gap function suppresses any contribution from the region $k\geqslant 2\mu$), this can be transformed into

$$\Pi_{44}^{00}(0)\simeq-3\,m_g^2\int_0^\mu\frac{d\xi}{\epsilon_\mathbf{k}^+}\left(1+\frac{\xi^2}{\mu^2}\right)\frac{\epsilon_\mathbf{k}^+ - \xi}{\epsilon_\mathbf{k}^+ + \xi}. \tag{117}$$

Neglecting the momentum dependence of the gap function, the remaining integral can be done introducing the variable

$$y\equiv\ln\left(\frac{\epsilon_\mathbf{k}^+ + \xi}{\phi_0}\right). \tag{118}$$

To leading order, the result is

$$\Pi_{44}^{00}(0)\simeq-\frac{3}{2}m_g^2. \tag{119}$$

The Debye mass (squared) is reduced by a factor 2 as compared to the value in the normal phase.

The limit $T\to T_c$ cannot be studied with Eq. (116), and one has to go back to Eq. (100a). It is obvious that one will reproduce the HDL result (53).

(ii) $\mu=0$, $\nu=i$. With Eq. (100b), and the same approximations as before, one obtains

$$\Pi_{44}^{0i}(0) \simeq -\frac{g^2 N_f}{2\pi^2}\int_0^\infty dk\, k^2 \int \frac{d\Omega}{4\pi}\hat{k}^i \left\{ \frac{1-n_{\mathbf{k}}^+}{\epsilon_{\mathbf{k}}^+ - \xi}[N_F^+(k) - N(\epsilon_{\mathbf{k}}^+)] + \frac{n_{\mathbf{k}}^+}{\epsilon_{\mathbf{k}}^+ + \xi}[1 - N_F^+(k) - N(\epsilon_{\mathbf{k}}^+)] \right\} \equiv 0, \tag{120}$$

by symmetry.

(iii) $\mu = i$, $\nu = j$. From Eq. (100c) one derives under the same approximations

$$\Pi_{44}^{ij}(0) \simeq -\delta^{ij}\frac{g^2 N_f}{6\pi^2}\int_0^\infty dk\, k^2 \times \left\{ \frac{1-n_{\mathbf{k}}^+}{\epsilon_{\mathbf{k}}^+ - \xi}[N_F^+(k) - N(\epsilon_{\mathbf{k}}^+)] + \frac{n_{\mathbf{k}}^+}{\epsilon_{\mathbf{k}}^+ + \xi}[1 - N_F^+(k) - N(\epsilon_{\mathbf{k}}^+)] + \frac{1}{k}[1 - N_F^+(k)] + 2\,\frac{1-n_{\mathbf{k}}^+}{\epsilon_{\mathbf{k}}^+ + k + \mu}[1 - N(\epsilon_{\mathbf{k}}^+)] - 2\,\frac{n_{\mathbf{k}}^+}{\epsilon_{\mathbf{k}}^+ - k - \mu}N(\epsilon_{\mathbf{k}}^+) - \frac{2}{k} \right\}, \tag{121}$$

where the last term is a vacuum subtraction.

At $T=0$, the integral over the first two terms in the integrand has already been computed for $\Pi_{44}^{00}(0)$, with the result (119). This is cancelled by a part of the vacuum subtraction. The remainder is

$$\Pi_{44}^{ij}(0) \simeq \delta^{ij}\frac{g^2 N_f}{6\pi^2}\int_0^\infty dk\, \frac{k}{\epsilon_{\mathbf{k}}^+}\,\frac{\mu\,(\epsilon_{\mathbf{k}}^+ - \xi) + [\phi_{\mathbf{k}}^+]^2}{\epsilon_{\mathbf{k}}^+ + \xi + 2\mu}. \tag{122}$$

Because the momentum dependence of the gap function suppresses the contribution from momenta far from the Fermi surface, the integral can be restricted to the region $0 \leqslant k \leqslant 2\,\mu$. To leading order, one may neglect $[\phi_{\mathbf{k}}^+]^2$ in the numerator. [The respective contribution is of order $\phi_0^2 \ln(2\mu/\phi_0)$.] Then, introduce the integration variable $z = \epsilon_{\mathbf{k}}^+ - k + \mu$. Neglecting the momentum dependence of the gap function, as well as terms of order $[\phi_{\mathbf{k}}^+]^2$, one obtains

$$\Pi_{44}^{ij}(0) \simeq \delta^{ij}\frac{g^2 N_f}{12\pi^2}\mu \int_0^{2\mu} dz \left(1 - \frac{z}{2\mu}\right) = \delta^{ij}\frac{m_g^2}{2}. \tag{123}$$

The limit $T \to T_c$ is not well-defined for Eq. (121); using Eq. (100c) it is, however, straightforward to show that $\Pi_{44}^{ij}(0) \to \Pi_0^{ij}(0)$, as expected.

C. Gluons with color 5 and 7

In the limit $p_0 = 0$, $p \to 0$, i.e., $\mathbf{k}_2 \to \mathbf{k}_1$, it is obvious from comparing Eqs. (96b) and (96c) that

$$\Pi_{44}^{\mu\nu}(0) \equiv \Pi_{55}^{\mu\nu}(0), \tag{124}$$

hence, the results from the previous subsection can be carried over.

D. Gluons with color 8

(i) $\mu = \nu = 0$. From Eq. (102a) one obtains with the approximations (103)

$$\tilde{\Pi}^{00}(0) \simeq \frac{g^2 N_f}{2\pi^2}\int_0^\infty dk\, k^2 \left\{ \frac{dn_{\mathbf{k}}^+}{dk}[1 - 2N(\epsilon_{\mathbf{k}}^+)] + \frac{dN(\epsilon_{\mathbf{k}}^+)}{dk}(1 - 2n_{\mathbf{k}}^+) \right\}. \tag{125}$$

The integrand is vanishingly small except close to the Fermi surface. One can therefore restrict the k integration to the range $0 \leqslant k \leqslant 2\mu$. Then, introducing ξ as integration variable and using the symmetry of the integrand around $\xi = 0$,

$$\tilde{\Pi}^{00}(0) \simeq -3\, m_g^2 \int_0^\mu d\xi \frac{d}{d\xi}\left[\frac{\xi}{\epsilon_{\mathbf{k}}^+}\tanh\left(\frac{\epsilon_{\mathbf{k}}^+}{2T}\right)\right], \tag{126}$$

where higher order terms ($\sim \xi^2/\mu^2$) in the integrand have been neglected. The remaining integral is unity (remember that $\mu \gg T$), and the final result is

$$\tilde{\Pi}^{00}(0) \simeq -3\, m_g^2. \tag{127}$$

Note that this result is independent of the temperature. One concludes that

$$\Pi_{88}^{00}(0) \equiv \frac{2}{3}\Pi_0{}^{00}(0) + \frac{1}{3}\tilde{\Pi}^{00}(0) \equiv -3\, m_g^2 \tag{128}$$

does not change with temperature in the superconducting phase; it always has the same value as in the normal phase.

(ii) $\mu = 0$, $\nu = i$. On account of Eqs. (102b) and (111),

$$\tilde{\Pi}^{0i}(0,\mathbf{p}) \simeq 0. \tag{129}$$

Consequently, also $\Pi_{88}^{0i}(0) \simeq 0$.

(iii) $\mu = i$, $\nu = j$. For $\tilde{\Pi}^{ij}(0)$ one derives from Eq. (102c) with the standard approximations

$$\tilde{\Pi}^{ij}(0) \simeq -\delta^{ij}\frac{g^2 N_f}{6\pi^2}\int_0^\infty dk\, k^2 \left\{ -\frac{dN(\epsilon_{\mathbf{k}}^+)}{d\epsilon_{\mathbf{k}}^+} + \frac{4\,[1 - N(\epsilon_{\mathbf{k}}^+)]\,(1 - n_{\mathbf{k}}^+)}{\epsilon_{\mathbf{k}}^+ + k + \mu} - \frac{4\, N(\epsilon_{\mathbf{k}}^+)\, n_{\mathbf{k}}^+}{\epsilon_{\mathbf{k}}^+ - k - \mu} - \frac{2}{k} \right\}, \tag{130}$$

where the last term is a vacuum subtraction.

At $T=0$, Eq. (130) becomes twice the integral in Eq. (122), hence

$$\tilde{\Pi}^{ij}(0) \simeq \delta^{ij}\, m_g^2. \tag{131}$$

As a consequence,

$$\Pi_{88}^{ij}(0) \simeq \delta^{ij} \frac{m_g^2}{3}. \tag{132}$$

As $T \to T_c$, an integration by parts shows that $\tilde{\Pi}^{ij}(0) \to 0$, as it should be. Consequently, also $\Pi_{88}^{ij}(0) \to 0$.

This concludes the discussion of Debye screening and the Meissner effect. In the next section, it will be demonstrated that for momenta $p \gg \phi_0$, i.e., when the gluon momentum is large enough to resolve the quarks in a Cooper pair, the gluon self-energy approaches the value in the normal phase.

VI. NONZERO GLUON MOMENTUM

In this section, the gluon self-energy will be computed in the static limit, but for gluon momenta $\phi_0 \ll p \ll \mu$. In the condensed matter literature, this limit is known as the Pippard limit [4]. The actual calculation follows closely that for ordinary superconductors (see for instance [4]). It will be convenient to consider the difference between the self energies in the superconducting and normal phases,

$$\delta\Pi \equiv \Pi - \Pi_0. \tag{133}$$

For large gluon momenta, effects from the pairing of quarks have to vanish, as the gluon wave length is short enough to resolve individual quarks in a Cooper pair. Consequently, the Debye mass for gluons of color 1, 2, and 3 can no longer vanish, but must approach the value in the normal phase. Simultaneously, for gluons of color 8 the Meissner effect has to vanish. These are the two cases studied in this section.

Of course, also the electric and magnetic masses of gluons with colors 4, 5, 6, and 7 have to approach their values in the normal phase. I was, however, not able to derive simple analytical expressions for the self-energy of these gluons in the limit $\phi_0 \gg p \gg \mu$. An explicit numerical study will be deferred to the future.

First note that for $p \ll \mu$, $k \sim \mu$,

$$k_{1,2} \simeq k \pm \frac{\hat{\mathbf{k}}\cdot\mathbf{p}}{2}. \tag{134}$$

This then leads to the same expressions (46a)–(46c) for the spin traces as in the HDL limit. As in the previous section, quasi-antiparticles will be treated as real antiparticles, cf. Eq. (103). Furthermore, for the sake of notational convenience, let us introduce

$$\xi_\pm \equiv \xi \pm \frac{\hat{\mathbf{k}}\cdot\mathbf{p}}{2}, \quad \epsilon_\pm \equiv \epsilon^+_{\mathbf{k}_{1,2}}, \quad \phi_\pm \equiv \phi^+(\epsilon_\pm),$$
$$n_\pm \equiv n^+_{\mathbf{k}_{1,2}}, \quad N_\pm \equiv N(\epsilon_\pm). \tag{135}$$

A. Electric gluons of color 1, 2, and 3

Writing $N_\pm = [1 - \tanh(\epsilon_\pm/2T)]/2$, the self-energy of electric gluons of colors 1, 2, and 3 is from Eq. (99a)

$$\Pi_{11}^{00}(0,\mathbf{p}) \simeq -\frac{g^2 N_f}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \times \left\{ \frac{1}{\epsilon_+ + \epsilon_-} \left[\tanh\left(\frac{\epsilon_+}{2T}\right) + \tanh\left(\frac{\epsilon_-}{2T}\right) \right] \times \frac{1}{2}\left(1 - \frac{\xi_+\xi_- + \phi_+\phi_-}{\epsilon_+\epsilon_-}\right) + \frac{1}{\epsilon_+ - \epsilon_-}\left[\tanh\left(\frac{\epsilon_+}{2T}\right) - \tanh\left(\frac{\epsilon_-}{2T}\right)\right] \frac{1}{2}\left(1 + \frac{\xi_+\xi_- + \phi_+\phi_-}{\epsilon_+\epsilon_-}\right)\right\}, \tag{136}$$

where terms of order p^2/k^2 have been neglected. The self-energy in the normal phase can be obtained either from Eq. (44), for $p_0 = 0$ and with the approximations (103), or directly from Eq. (136) in the limit $\phi_\pm \to 0$:

$$\Pi_0{}^{00}(0,\mathbf{p}) \simeq -\frac{g^2 N_f}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\xi_+ - \xi_-} \times \left[\tanh\left(\frac{\xi_+}{2T}\right) - \tanh\left(\frac{\xi_-}{2T}\right) \right]. \tag{137}$$

Now consider the difference $\delta\Pi_{11}^{00}(0,\mathbf{p})$ between Eqs. (136) and (137). As the main contribution to the integral over $\mathbf{k}$ comes from the region around the Fermi surface, it is admissible to neglect the momentum dependence of the gap function, $\phi_+ \simeq \phi_- \equiv \phi$. Then one rearranges the integrand to separate terms of the form

$$\frac{1}{\xi_+ - \xi_-}\left[\frac{\xi_\pm}{\epsilon_\pm} \tanh\left(\frac{\epsilon_\pm}{2T}\right) - \tanh\left(\frac{\xi_\pm}{2T}\right) \right]. \tag{138}$$

As argued in [4], these terms vanish by symmetry when integrating over ξ. (A careful analysis shows that this is correct to leading order in ϕ/p.) The result is

$$\delta\Pi_{11}^{00}(0,\mathbf{p}) \simeq -\frac{g^2 N_f}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\phi^2}{\xi\, \hat{\mathbf{k}}\cdot\mathbf{p}} \times \left[\frac{1}{\epsilon_+} \tanh\left(\frac{\epsilon_+}{2T}\right) - \frac{1}{\epsilon_-}\tanh\left(\frac{\epsilon_-}{2T}\right) \right]. \tag{139}$$

As the integrand peaks at the Fermi surface, $\xi \simeq 0$, and for $\hat{\mathbf{k}}\cdot\mathbf{p} \simeq 0$, one can approximate the hyperbolic tangents by $\tanh(\epsilon_\pm/2T) \sim \tanh(\phi/2T)$, and obtains to leading order

$$\delta\Pi_{11}^{00}(0,\mathbf{p}) \simeq -3\, m_g^2 \frac{\phi}{p} \tanh\left(\frac{\phi}{2T}\right) \int_0^{\mu/\phi} \frac{dx}{x} \int_0^{p/2\phi} \frac{dy}{y} \times \left(\frac{1}{\sqrt{(x+y)^2+1}} - \frac{1}{\sqrt{(x-y)^2+1}} \right), \tag{140}$$

where $x \equiv \xi/\phi$, $y \equiv \hat{\mathbf{k}}\cdot\mathbf{p}/(2\phi)$. The y integral can be done exactly. In the limit $\mu \gg p \gg \phi$,

$$\delta\Pi_{11}^{00}(0,\mathbf{p})\simeq 3\, m_g^2 \frac{\phi}{p}\tanh\left(\frac{\phi}{2T}\right)\int_0^\infty du\,\frac{2u}{\sinh u}$$

$$\equiv 3\, m_g^2 \frac{\pi^2}{2}\,\frac{\phi}{p}\tanh\left(\frac{\phi}{2T}\right). \tag{141}$$

The self-energy in the normal phase is approximately constant for momenta $p \ll \mu$, such that

$$\Pi_{11}^{00}(0,\mathbf{p})\simeq -3\, m_g^2\left[1-\frac{\pi^2}{2}\,\frac{\phi}{p}\tanh\left(\frac{\phi}{2T}\right)\right]. \tag{142}$$

This shows that the absolute value of the self-energy in the superconducting phase is reduced as compared to the normal phase. For increasing p/ϕ, the correction becomes smaller, such that electric fields for adjoint colors 1, 2, and 3 are screened over an only slightly longer distance than in the normal phase. In this case, the gluons ''see'' the individual fundamental color charges inside the Cooper pairs.

For decreasing p/ϕ, however, the correction becomes larger. This is in agreement with the results of Sec. V, where the self-energy of gluons with colors 1, 2, and 3 was found to vanish in the limit $p\rightarrow 0$, i.e., when the gluon momentum is too small to resolve individual quarks inside a Cooper pair. Although strictly valid only for $p \gg \phi$, by extrapolating Eq. (142) to $p\sim\phi$ one would conclude that, at $T=0$, this happens once p is smaller than $\simeq 5\,\phi_0$.

B. Magnetic gluons of color 8

For magnetic gluons, one derives from Eq. (102c)

$$\tilde{\Pi}^{ij}(0,\mathbf{p})\simeq -\frac{g^2 N_f}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\hat{k}^i\hat{k}^j\left\{\frac{1}{\xi_+-\xi_-}\left[\frac{\xi_+}{\epsilon_+}\tanh\left(\frac{\epsilon_+}{2T}\right)-\frac{\xi_-}{\epsilon_-}\tanh\left(\frac{\epsilon_-}{2T}\right)\right]\right.\right.$$
$$\left.+\frac{\phi^2}{\xi\,\hat{\mathbf{k}}\cdot\mathbf{p}}\left[\frac{1}{\epsilon_+}\tanh\left(\frac{\epsilon_+}{2T}\right)-\frac{1}{\epsilon_-}\tanh\left(\frac{\epsilon_-}{2T}\right)\right]\right\}+(\delta^{ij}-\hat{k}^i\hat{k}^j)\,\frac{1}{2k}\left\{2+\frac{\xi_+}{\epsilon_+}\tanh\left(\frac{\epsilon_+}{2T}\right)+\frac{\xi_-}{\epsilon_-}\tanh\left(\frac{\epsilon_-}{2T}\right)\right.$$
$$\left.\left.-\frac{\phi^2}{2\mu}\left[\frac{1}{\epsilon_+}\tanh\left(\frac{\epsilon_+}{2T}\right)+\frac{1}{\epsilon_-}\tanh\left(\frac{\epsilon_-}{2T}\right)\right]\right\}\right). \tag{143}$$

Here, the momentum dependence of the gap function was neglected, $\phi_\pm\simeq\phi$. Moreover, in denominators which contain terms $\sim\mu^2$, $\epsilon_\pm^2$ was approximated by $\xi_\pm^2$.

In the normal phase, the corresponding expression reads

$$\Pi_0^{ij}(0,\mathbf{p})\simeq -\frac{g^2 N_f}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left\{\hat{k}^i\hat{k}^j\frac{1}{\xi_+-\xi_-}\left[\tanh\left(\frac{\xi_+}{2T}\right)-\tanh\left(\frac{\xi_-}{2T}\right)\right]+(\delta^{ij}-\hat{k}^i\hat{k}^j)\,\frac{1}{2k}\left[2+\tanh\left(\frac{\xi_+}{2T}\right)+\tanh\left(\frac{\xi_-}{2T}\right)\right]\right\}. \tag{144}$$

In the difference $\delta\tilde{\Pi}^{ij}$, there are again terms like Eq. (138), which vanish by symmetry arguments. There is also a term $\sim\phi^2/(4\mu k)$ which is of higher order and thus can be neglected. The remainder can be written as

$$\delta\tilde{\Pi}^{ij}(0,\mathbf{p})\simeq -3\, m_g^2\frac{\phi}{p}\int_0^{\mu/\phi}\frac{dx}{x}\int_0^{p/2\phi}\frac{dy}{y}\;\tanh\left(\frac{\phi\sqrt{y^2+1}}{2T}\right)\left(\frac{1}{\sqrt{(x+y)^2+1}}-\frac{1}{\sqrt{(x-y)^2+1}}\right)\int_0^{2\pi}\frac{d\varphi}{2\pi}\hat{k}^i\hat{k}^j. \tag{145}$$

As before, $x\equiv\xi/\phi$, $y\equiv\hat{\mathbf{k}}\cdot\mathbf{p}/(2\phi)$. Since the x integral is dominated by the region around the Fermi surface, $x\simeq 0$, I have set $x=0$ in the argument of the hyperbolic tangents.

For $i\neq j$, the integration over the polar angle φ vanishes, thus $\delta\tilde{\Pi}^{ij}$ is diagonal. However, not all diagonal elements are equal. Let $\mathbf{p}=(0,0,p)$. Then $\hat{k}_x^2=[1-(2\phi y/p)^2]\cos^2\varphi$, $\hat{k}_y^2=[1-(2\phi y/p)^2]\sin^2\varphi$, and the transverse components of $\delta\tilde{\Pi}^{ij}$ are

$$\delta\tilde{\Pi}^{xx}(0,\mathbf{p})\equiv\delta\tilde{\Pi}^{yy}(0,\mathbf{p})\simeq m_g^2\frac{3\pi^2}{4}\,\frac{\phi}{p}\;\tanh\left(\frac{\phi}{2T}\right). \tag{146}$$

To obtain this result, I have used the fact that the y integration is dominated by the region $y\simeq 0$, and consequently have set $y=0$ in the hyperbolic tangents as well as in $\hat{k}_{x,y}^2$. The remaining integral is then the same as in Eq. (140).

The longitudinal component can be shown to be of higher order in ϕ/p, such that to leading order,

$$\delta\tilde{\Pi}^{zz}(0,\mathbf{p})\simeq 0. \tag{147}$$

This result is not unexpected: the self-energy for gluons in the normal phase is transverse, $\Pi_0{}^{ij}(0,\mathbf{p})\simeq(\delta^{ij}-\hat{p}^i\hat{p}^j)\,p^2 m_g^2/(12\mu^2)$. [Note that this expressions is of order $g^2p^2 \ll g^2\mu^2$, and thus not in contradiction to the HDL result (61).] Equations (146) and (147) now combine to give a transverse self-energy for the eighth gluon, too,

$$\Pi_{88}^{ij}(0,\mathbf{p}) \simeq (\delta^{ij} - \hat{p}^i \hat{p}^j)\, m_g^2 \left[\frac{p^2}{12\,\mu^2} + \frac{\pi^2}{4}\,\frac{\phi}{p} \tanh\left(\frac{\phi}{2T}\right) \right]. \tag{148}$$

VII. SUMMARY, CONCLUSIONS, AND OUTLOOK

In color-superconducting quark matter with $N_f = 2$ degenerate quark flavors, the condensate can be oriented in the (anti-)3 direction in fundamental color space by means of a global color rotation. Then, only quarks with fundamental colors 1 and 2 form Cooper pairs, while quarks of the third fundamental color remain unpaired, and act as a background to neutralize the color-charged condensate. Since the unpaired quarks carry the same color charge, two of them are in the (repulsive) sextet representation of $SU(3)_c$. Consequently, they do not form Cooper pairs and the system is stable.

The condensate breaks the $SU(3)_c$ color symmetry to $SU(2)_c$. With the above color choice, the generators of the unbroken $SU(2)_c$ subgroup are the $SU(3)_c$ generators T^1, T^2, and T^3, with $T^a = \lambda^a/2$ and the standard convention for the Gell-Mann matrices λ^a. The gluons corresponding to the remaining generators T^4 through T^8 all receive a mass via the Anderson-Higgs mechanism.

What are the expected values for these masses? The effective Lagrangian for the low-energy excitations of the condensate fields minimally coupled to gauge fields has the kinetic term [19]

$$\mathcal{L}_{\text{eff}}^{\text{kin}} = \alpha_{\text{e}}\,(D_0\Phi)^\dagger D^0\Phi + \alpha_{\text{m}}\,(D_i\Phi)^\dagger D^i\Phi. \tag{149}$$

The presence of a heat and particle bath at nonzero T and/or μ breaks Lorentz invariance, so that the coefficient α_{e} of the part containing the time derivatives can in principle be different from the one of the part containing the spatial derivatives, α_{m}.

For a two-flavor color-superconductor, Φ is an $SU(3)_c$ (anti-)triplet, $\Phi \equiv (\Phi_1, \Phi_2, \Phi_3)^T$ [7]. Consequently, the covariant derivative is $D_\mu = \partial_\mu - igA_\mu^a T^a$, with the generators T^a being in the fundamental representation. If Φ attains a non-vanishing expectation value $\langle \Phi \rangle = (0,0,\phi_0)^T$, $\phi_0 \in \mathbf{R}$, this generates a mass term for the gluon fields of the form

$$\begin{aligned}\mathcal{L}_1^{\text{M}} &= g^2\,\phi_0^2 (\alpha_{\text{e}} A_0^a A_b^0 + \alpha_{\text{m}} A_i^a A_a^i)\,\delta_{3i}\,T_{ij}^a\,T_{jk}^b\,\delta_{k3} \\ &\equiv g^2\,\phi_0^2 \left[\frac{1}{4} \sum_{a=4}^{7} (\alpha_{\text{e}} A_0^a A_a^0 + \alpha_{\text{m}} A_i^a A_a^i) \right. \\ &\quad \left. + \frac{1}{3}(\alpha_{\text{e}} A_0^8 A_8^0 + \alpha_{\text{m}} A_i^8 A_8^i) \right]. \end{aligned} \tag{150}$$

The expected electric and magnetic gluon masses are

TABLE I. Results for the Debye and Meissner masses in a two-flavor color superconductor.

Gluon color	$-\Pi_{aa}^{00}(0)$		$\Pi_{aa}^{ii}(0)$	
a	$T=0$	$T \geq T_c$	$T=0$	$T \geq T_c$
1–3	0	$3\,m_g^2$	0	0
4–7	$\frac{3}{2}m_g^2$	$3\,m_g^2$	$\frac{1}{2}\,m_g^2$	0
8	$3\,m_g^2$	$3\,m_g^2$	$\frac{1}{3}\,m_g^2$	0

$$\begin{aligned} M_{\text{e,m}}^1 &= M_{\text{e,m}}^2 = M_{\text{e,m}}^3 = 0, \\ M_{\text{e,m}}^4 &= M_{\text{e,m}}^5 = M_{\text{e,m}}^6 = M_{\text{e,m}}^7 = \sqrt{\frac{\alpha_{\text{e,m}}}{2}}\,g\,\phi_0, \\ M_{\text{e,m}}^8 &= \sqrt{\frac{2\,\alpha_{\text{e,m}}}{3}}\,g\,\phi_0, \end{aligned} \tag{151}$$

such that the ratio

$$R_{\text{e,m}} \equiv \left(\frac{M_{\text{e,m}}^8}{M_{\text{e,m}}^4} \right)^2 = 4/3. \tag{152}$$

In this work, the gluon self-energy in a $N_f = 2$ color superconductor has been derived. Due to the pattern of symmetry breaking, one has to study the individual gluon colors separately. The central result are Eqs. (99a)–(102c). Various limits of these expressions are of interest. Here, the self-energy was computed in the static, homogeneous limit, $p_0 = 0$, $p \to 0$, which yields the Debye mass for electric and the Meissner mass for magnetic gluons. The main results are summarized in Table I.

For the three gluons of the unbroken $SU(2)_c$ subgroup (gluon colors 1, 2, and 3), the Debye mass as well as the Meissner mass vanish. While this is in agreement with Eq. (151), it is at first physically unclear, and therefore quite surprising, why gluon fields with colors 1, 2, and 3 are not screened. To explain this, I argued as follows. Gluons with adjoint colors 1, 2, and 3 couple to fundamental colors 1 and 2. At $T = 0$, however, all quarks with these color charges are bound in Cooper pairs which have fundamental color (anti-)3. Thus, these gluons cannot ''see'' the quark charges, and hence are unscreened. At nonzero T, quasiparticles are thermally excited. They have the ''right'' fundamental color (1 and 2) to screen gluon fields with adjoint colors 1, 2, and 3, and consequently lead to screening and a nonzero Debye mass. At $T = T_c$, when the condensate melts, the Debye mass assumes its standard value in the normal phase.

Of course, at $T = 0$ the gluon self-energy vanishes only in the zero-energy, zero-momentum limit, since then the gluon field cannot resolve individual quarks inside the Cooper pair. For large gluon momentum $p \gg \phi_0$, electric gluon fields are screened; the self-energy is the same as in the normal phase, up to a correction of order $\sim m_g^2\,\phi_0/p$, as computed in Sec. VI A.

The gluons corresponding to the broken generators of $SU(3)_c$ all attain a mass through the Anderson-Higgs mechanism. While the Debye mass for electric gluons of color 8 is the same as in the normal phase, the Debye mass squared for colors 4 through 7 is only half as large. As T approaches T_c, however, the melting of the condensate leads to an increase of the Debye mass to its standard value. At zero temperature, the ratio of the Debye masses squared of gluon color 8 and 4 is $R_e \equiv \Pi_{88}^{00}(0)/\Pi_{44}^{00}(0) = 2$.

The Meissner mass squared for gluons of color 8 is 1/3 of the gluon mass squared, m_g^2, while that for gluons of colors 4 through 7 is 1/2 of the gluon mass squared. The Meissner effect vanishes as T approaches T_c, or when the gluon momentum $p \gg \phi_0$, as computed in Sec. VI B. The ratio of the Meissner masses squared of gluon color 4 and 8 is $R_m \equiv \Pi_{88}^{ii}(0)/\Pi_{44}^{ii}(0) = 2/3$.

Both R_e and R_m differ from the expectation (152). What is the origin of this discrepancy? The kinetic term (150) is not the only possible invariant in an effective Lagrangian, where the condensate fields are minimally coupled to the gauge fields. Another possibility is the term [20]

$$\mathcal{L}'_{\rm eff} = \beta_e (\Phi^\dagger D_0 \Phi)^\dagger \Phi^\dagger D^0 \Phi + \beta_m (\Phi^\dagger D_i \Phi)^\dagger \Phi^\dagger D^i \Phi, \tag{153}$$

which has mass dimension six [consequently, $\beta_{e,m}$ have dimension (mass)$^{-2}$]. Note that in the nonlinear version of the effective theory [21], where the modulus of Φ does not change, only the phase, this term is identical to the standard kinetic term (150).

Upon condensation, $\langle \Phi \rangle = (0,0,\phi_0)^T$, the term (153) contributes to the mass of the eighth gluon,

$$\mathcal{L}_2^M = g^2 \phi_0^4 \frac{1}{3} (\beta_e A_0^8 A_8^0 + \beta_m A_i^8 A_8^i). \tag{154}$$

With this term, one reproduces the zero-temperature magnetic masses given in Table I with the choice

$$\alpha_m \equiv \frac{m_g^2}{g^2 \phi_0^2} = \frac{N_f}{6\pi^2} \frac{\mu^2}{\phi_0^2}, \quad \beta_m \equiv -\frac{1}{2} \frac{m_g^2}{g^2 \phi_0^4} = -\frac{N_f}{12\pi^2} \frac{\mu^2}{\phi_0^4}. \tag{155}$$

Note that the prefactor of the kinetic term (150) has the $1/\phi_0^2$ behavior typical for effective theories of superconductivity [1,4,19]. To reproduce the electric masses, the coefficients α_e and β_e have to be chosen as

$$\alpha_e \equiv 3\, \alpha_m, \quad \beta_e = -3\, \beta_m. \tag{156}$$

The expressions (155) and (156) fix the prefactors of the kinetic term (150) and the higher-order term (153) in the effective low-energy theory of condensate fields coupled to gluons. Up to mass dimension four, the effective theory for an $SU(3)_c$ vector Φ has, apart from the gauge field part, two more terms which are invariant under $SU(3)_c$ transformations [7]: a mass term for the condensate field

$$\mathcal{L}_{\rm eff}^{\rm mass} = \mathcal{M}^2\, \Phi^\dagger \Phi, \tag{157}$$

and a quartic self interaction

$$\mathcal{L}_{\rm eff}^{\rm int} = \lambda\, (\Phi^\dagger \Phi)^2. \tag{158}$$

Work is in progress to determine the condensate mass $\mathcal{M}$ and the coupling constant λ [22].

What is the impact of these results for the solution of the gap equations? Remember that, after taking into account the color and flavor structure, the gap matrix in spinor space obeys the gap equation [6]

$$\Phi^+(K) = \frac{3}{4} g^2 \frac{T}{V} \sum_Q \left[\Delta_{11}^{\mu\nu}(K-Q) - \frac{1}{9} \Delta_{88}^{\mu\nu}(K-Q) \right] \times \gamma_\mu\, G_0^-(Q)\, \Phi^+(Q)\, G^+(Q)\, \gamma_\nu. \tag{159}$$

Previously [5,6,8,12], the gap equation was solved using the HDL propagator for both Δ_{11} and Δ_{88},

$$\Phi^+(K) = \frac{2}{3} g^2 \frac{T}{V} \sum_Q \Delta_{\rm HDL}^{\mu\nu}(K-Q) \times \gamma_\mu\, G_0^-(Q)\, \Phi^+(Q)\, G^+(Q)\, \gamma_\nu, \tag{160}$$

where $\Delta_{\rm HDL}^{-1} \equiv \Delta_0^{-1} + \Pi_0$. The integral on the right-hand side is dominated by gluons with small momenta, $K - Q \simeq 0$. In the HDL limit, however, static electric gluons are screened by the Debye mass, $\Pi_0{}^{00}(0) \simeq -3\, m_g^2$, cf. Eq. (53). Their contribution is therefore suppressed as compared to that of magnetic gluons which are not screened in the static limit, $\Pi_0{}^{ij}(0) \simeq 0$, cf. Eq. (61). The dominant contribution to the gap integral therefore comes from (nearly) static magnetic gluons. A careful analysis [5,6,8,12] shows that the gluon energy is not exactly zero, but $p_0 \simeq \phi_0$, while the gluon momentum is $p \simeq (m_g^2 \phi_0)^{1/3}$, and thus, in weak coupling, actually much larger than ϕ_0. The coefficient $c_{\rm QCD} = 3\pi^2/\sqrt{2}$ is determined by how many nearly static magnetic modes contribute, and by the precise form of the magnetic HDL propagator.

As shown in this paper, the gluon propagator in a two-flavor color superconductor is, at least in the static limit, $p_0 = 0$, and for small gluon momenta, $p \sim \phi_0$, drastically different from the HDL propagator. For instance, for gluon colors 1, 2, and 3, which constitute the main contribution to the gap equation (159), both magnetic *and* electric modes remain unscreened. For gluon color 8, previously unscreened static magnetic gluons attain a Meissner mass.

In order to assess the effect of these results on the solution of the gap equation, one needs to solve the gap equation with the full energy and momentum dependence of the gluon propagator in the superconducting phase, to decide which energies and momenta constitute the dominant contribution to the gap integral. If gluon energy and momentum are much larger than the zero-temperature gap, the impact will be rather small, because, as was shown in Sec. VI, the effect of the superconducting medium is only a small correction of order $O(\phi_0/p)$ to the standard HDL propagator. This might influence the prefactor of the exponential $\exp(-c_{\rm QCD}/g)$, but not $c_{\rm QCD}$ itself. On the other hand, if the dominant range of energies and momenta is $p_0, p \sim \phi_0$, the impact could be large and might even change c_{QCD}. A detailed analysis of this problem is under investigation [23].

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[1] D. Bailin and A. Love, Phys. Rep. **107**, 325 (1984).
[2] M. Alford, K. Rajagopal, and F. Wilczek, Phys. Lett. B **422**, 247 (1998); R. Rapp, T. Schäfer, E.V. Shuryak, and M. Velkovsky, Phys. Rev. Lett. **81**, 53 (1998); Ann. Phys. (N.Y.) **280**, 35 (2000); N. Evans, S.D.H. Hsu, and M. Schwetz, Nucl. Phys. **B551**, 275 (1999); Phys. Lett. B **449**, 281 (1999); J. Berges and K. Rajagopal, Nucl. Phys. **B538**, 215 (1999); T. Schäfer and F. Wilczek, Phys. Lett. B **450**, 325 (1999); G.W. Carter and D. Diakonov, Phys. Rev. D **60**, 016004 (1999); K. Langfeld and M. Rho, Nucl. Phys. **A660**, 475 (1999); M. Alford, J. Berges, and K. Rajagopal, Nucl. Phys. **B558**, 219 (1999).
[3] J.R. Schrieffer, *Theory of Superconductivity* (W.A. Benjamin, New York, 1964).
[4] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971); A.A. Abrikosov, L.P. Gorkov, and I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1963).
[5] R.D. Pisarski and D.H. Rischke, Phys. Rev. D **61**, 051501 (2000).
[6] R.D. Pisarski and D.H. Rischke, Phys. Rev. D **61**, 074017 (2000).
[7] R.D. Pisarski and D.H. Rischke, Phys. Rev. Lett. **83**, 37 (1999).
[8] D.T. Son, Phys. Rev. D **59**, 094019 (1999).
[9] R.D. Pisarski and D.H. Rischke, Phys. Rev. D **60**, 094013 (1999).
[10] D.K. Hong, Phys. Lett. B **473**, 118 (2000); hep-ph/9905523.
[11] D.K. Hong, V.A. Miransky, I.A. Shovkovy, and L.C.R. Wijewardhana, Phys. Rev. D **61**, 056001 (2000).
[12] T. Schäfer and F. Wilczek, Phys. Rev. D **60**, 114033 (1999).
[13] W.E. Brown, J.T. Liu, and H.-C. Ren, Phys. Rev. D **61**, 114012 (2000).
[14] S.D.H. Hsu and M. Schwetz, hep-ph/9908310.
[15] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[16] J.-P. Blaizot and J.-Y. Ollitrault, Phys. Rev. D **48**, 1390 (1993); H. Vija and M.H. Thoma, Phys. Lett. B **342**, 212 (1995); C. Manuel, Phys. Rev. D **53**, 5866 (1996).
[17] N. Evans, J. Hormuzdiar, S.D.H. Hsu, and M. Schwetz, hep-ph/9910313.
[18] R.D. Pisarski and D.H. Rischke, nucl-th/9906050.
[19] R.D. Pisarski, nucl-th/9912070.
[20] The existence of such a term was pointed out by I.A. Shovkovy.
[21] D.T. Son and M.A. Stephanov, Phys. Rev. D **61**, 074012 (2000).
[22] D.H. Rischke (work in progress).
[23] D.H. Rischke and D.T. Son (work in progress).